

The Derivatives of Sine and Cosine

Complete the following tables of values. Round off to at least 5 decimal places.

RADIANS	h	-0.1	-0.01	-0.001	0.001	0.01	0.1
	$y = \frac{\sin h}{h}$	0.99833	0.99998	0.9999998	0.9999998	0.99998	0.99833

degrees	h	-0.1	-0.01	-0.001	0.001	0.01	0.1
	$y = \frac{\sin h}{h}$	0.01745	0.01745	0.01745	0.01745	0.01745	0.01745

$\pi/180$

RADIANS	h	-0.1	-0.01	-0.001	0.001	0.01	0.1
	$y = \frac{\cos h - 1}{h}$	-0.04996	-0.00499	-0.000499	-0.000499	-0.00499	-0.04996

degrees	h	-0.1	-0.01	-0.001	0.001	0.01	0.1
	$y = \frac{\cos h - 1}{h}$	-0.000015	-0.0000015	-0.00000015	-0.00000015	-0.0000015	-0.000015

So in RADIANS, $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$ and $\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0$

Also recall that $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$.

For $f(x) = \sin x$ in RADIANS,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \right] \\
 &= \sin x \cdot 0 + \cos x \cdot 1 \\
 &= \cos x.
 \end{aligned}$$

To find the derivative of the cosine function, recall that

$$\cos x = \sin \left(x + \frac{\pi}{2} \right)$$

and that

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$$

Furthermore, the *chain rule* for the derivative of a composition of functions is

$$\frac{d}{dx} (f(g(x))) = f'(g(x)) \cdot g'(x).$$

As a result, in RADIANS

$$\begin{aligned} \frac{d}{dx}(\cos x) &= \frac{d}{dx} \left(\sin \left(x + \frac{\pi}{2} \right) \right) \\ &= \cos \left(x + \frac{\pi}{2} \right) \cdot 1 \\ &= \cos x \cos \frac{\pi}{2} - \sin x \sin \frac{\pi}{2} \\ &= \cos x \cdot 0 - \sin x \cdot 1 \\ &= -\sin x. \end{aligned}$$