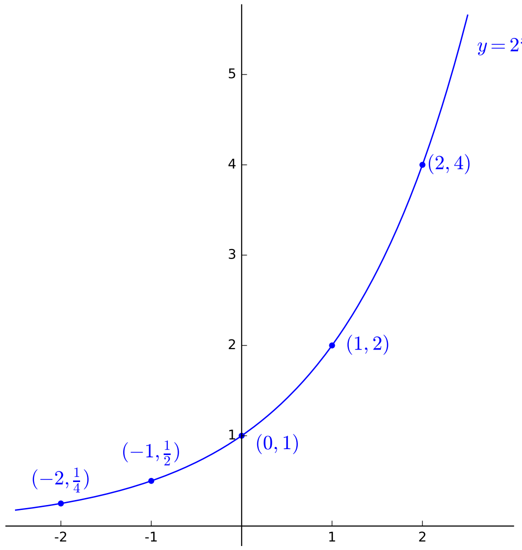
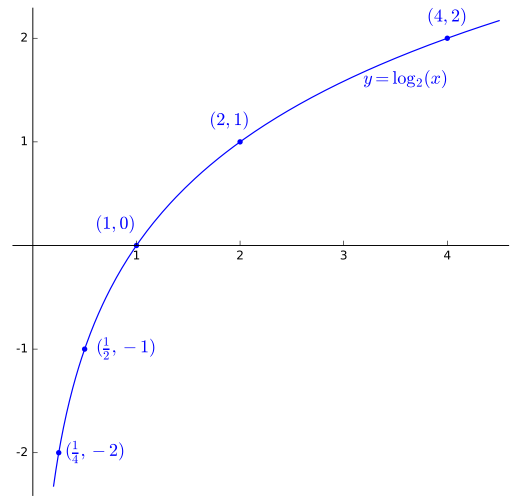
Review of Logarithmic and Exponential Functions

Example: Let's look at the exponential and log functions with **base 2**.

x	$y = 2^x$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$





Switching the coordinates of each point on the graph of $y = 2^x$ gives the graph of $y = \log_2(x)$.

function	domain	range
$y = 2^x$	$\mathbb{R}$	$(0, \infty)$
$y = \log_2 x$	$(0, \infty)$	$\mathbb{R}$

Logarithms are **powers**: $\log_2 8$ means "the power of 2 that makes 8", so that $\log_2 8 = 3$.

In general, for any base $a > 0$ and $a \neq 1$:

$$\boxed{y = a^x \quad \Leftrightarrow \quad x = \log_a y}$$

There are 2 special bases: 10 and $e \simeq 2.72$, the constant known as *Euler's number*

$$\log_{10} x = \log x \qquad \text{"common logarithm"}$$

$$\log_e x = \ln x \qquad \text{"natural logarithm"}$$

Derivatives of Logarithmic and Exponential Functions

1. Let's start by finding the derivative of $f(x) = \ln x$ using the limit definition of the derivative. We will need the following:

   - the definition $e = \lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u$
   - the 2 log properties $\log_b M - \log_b N = \log_b\left(\frac{M}{N}\right)$ and $\log_b(M^n) = n\log_b M$
   - $\ln e = 1$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} \\ &= \lim_{h\to 0}\frac{\ln(\frac{x+h}{x})}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\cdot\ln\left(1+\frac{h}{x}\right) \\ &= \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}} \end{aligned}$$

   Now we need a change of variable: let $u = \frac{x}{h}$. Then $\frac{h}{x} = \frac{1}{u}$, $\frac{1}{h} = \frac{1}{x}\cdot u$, and $h \to 0 \Leftrightarrow u \to \infty$. So

$$\begin{aligned} f'(x) &= \lim_{u\to\infty}\ln\left(1+\frac{1}{u}\right)^{\frac{1}{x}\cdot u} \\ &= \lim_{u\to\infty}\frac{1}{x}\cdot\ln\left(1+\frac{1}{u}\right)^u \\ &= \frac{1}{x}\cdot\ln e \\ &= \frac{1}{x} \end{aligned}$$

2. For the derivative of $f(x) = \log_a x$ for any base a, recall the change of base formula $\log_a x = \dfrac{\log_b x}{\log_b a}$. So

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\cdot\frac{d}{dx}(\ln x) = \frac{1}{\ln a}\cdot\frac{1}{x} = \frac{1}{x\ln a}$$

3. To find $\dfrac{dy}{dx}$ for $y = a^x$, we use implicit differentiation:

$$\begin{aligned} y &= a^x \\ \log_a y &= x \\ \frac{d}{dx}(\log_a y) &= \frac{d}{dx}(x) \\ \frac{1}{y\ln a}\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y\ln a = a^x\ln a \end{aligned}$$