31.2 and 31.4 Mixed DEs SOLUTIONS

1. $2\dfrac{dy}{dx} = \dfrac{y(x+1)}{x}$ separable

$$\frac{2}{y}\,dy = \frac{(x+1)}{x}\,dx$$

$$2\int \frac{1}{y}\,dy = \int \left(1 + \frac{1}{x}\right)dx$$

$$\boxed{2\ln y = x + \ln x + C}$$

implicit solution – fine to stop here

to find the explicit solution: $2\ln y = x + \ln x + C_1$

$$\ln y = \tfrac{1}{2}x + \tfrac{1}{2}\ln x + \boxed{\tfrac{1}{2}C_1}$$ let $C_2 = \frac{1}{2}C_1$

$$y = e^{\frac{1}{2}x + \frac{1}{2}\ln x + C_2}$$

$$y = e^{\frac{1}{2}x}\, e^{\frac{1}{2}\ln x}\, \boxed{e^{C_2}}$$ let $C = e^{C_2}$

$$y = C e^{\frac{1}{2}x} e^{\ln\sqrt{x}}$$

$$y = C\sqrt{x}\,e^{\frac{1}{2}x}$$

2. $y' + y\tan x = -\sin x$ in linear first-order form

$P(x) = \tan x$, $Q(x) = -\sin x$

$$e^{\int P(x)\,dx} = e^{\int \tan x\,dx} = e^{\ln(\sec x)} = \sec x$$

$$\sec x\left(y' + y\tan x\right) = \left(-\sin x\right)\sec x$$

$$\underbrace{\sec x}_{u}\,\underbrace{y'}_{v'} + \underbrace{y}_{v}\,\underbrace{\tan x\sec x}_{u'} = -\sin x\sec x$$

$$\frac{d}{dx}\left(\underbrace{\sec x}_{u}\cdot\underbrace{y}_{v}\right) = -\sin x\sec x$$

$$d\left(\sec x\cdot y\right) = -\sin x\sec x\,dx$$

$$\int d\left(\sec x\cdot y\right) = \int -\sin x\sec x\,dx$$

it's fine to skip these steps

$$\sec x\cdot y = \int -\sin x\sec x\,dx$$

$$y\sec x = -\int \frac{\sin x}{\cos x}\,dx$$

$$y\sec x = -\int \tan x\,dx$$

$$\boxed{y\sec x = -\ln(\sec x) + C}$$ implicit solution

$$y = \frac{-\ln(\sec x) + C}{\sec x} = \cos x\left(-\ln(\sec x) + C\right)$$ explicit solution

3. $y' = x^2 y + 3x^2$

$$\frac{dy}{dx} = x^2(y+3)$$

separable (but would also work as linear first-order with $P(x) = -x^2$)

$$\frac{1}{y+3} = x^2 dx$$

$$\int \frac{1}{y+3} dy = \int x^2 dx$$

$$\boxed{\ln(y+3) = \frac{1}{3}x^3 + C}$$

implicit solution - fine to stop here

to find the explicit solution: $\ln(y+3) = \frac{1}{3}x^3 + C_1$

$$y+3 = e^{\frac{1}{3}x^3 + C_1}$$

$$y+3 = e^{\frac{1}{3}x^3}\boxed{e^{C_1}} \quad \text{let } C = e^{C_1}$$

$$y+3 = Ce^{\frac{1}{3}x^3}$$

$$y = Ce^{\frac{1}{3}x^3} - 3$$

explicit solution

4. $y' = 2y + 2e^{2x}$

$$\frac{dy}{dx} \underbrace{- 2}_{P(x)} y = \underbrace{2e^{2x}}_{Q(x)}$$

$$e^{\int P(x)dx} = e^{\int -2dx} = e^{-2x}$$

$$e^{-2x}\left(\frac{dy}{dx} - 2y\right) = \left(2e^{2x}\right) e^{-2x}$$

$$\frac{d}{dx}\left(e^{-2x} y\right) = 2$$

$$\int d\left(e^{-2x} y\right) = \int 2 dx$$

$$\boxed{e^{-2x} y = 2x + C}$$ implicit solution

$$y = \frac{2x + C}{e^{-2x}} = e^{2x}(2x + C)$$ explicit solution

5. $xyy' + \sqrt{1+y^2} = 0$

$xyy' = -\sqrt{1+y^2}$

$\frac{dy}{dx} = \frac{-\sqrt{1+y^2}}{xy}$ separable

$\frac{-y}{\sqrt{1+y^2}}\,dy = \frac{1}{x}\,dx$

$-\int y\left(1+y^2\right)^{-1/2}\,dy = \int \frac{1}{x}\,dx$

let $u = 1+y^2$
$du = 2y\,dy$
$\frac{1}{2}du = y\,dy$

$-\frac{1}{2}\int u^{-1/2}\,du = \ln x + C_1$

$-\frac{1}{2}\cdot\frac{2}{1}\,u^{1/2} = \ln x + C_1$

$\left(1+y^2\right)^{1/2} = -\ln x \boxed{-C_1}$ let $C = -C_1$

$\boxed{\left(1+y^2\right)^{1/2} = -\ln x + C}$ implicit solution

to find the explicit solution: $\left(\left(1+y^2\right)^{1/2}\right)^2 = \left(-\ln x + C\right)^2$

$1+y^2 = \left(-\ln x + C\right)^2$

$y^2 = \left(-\ln x + C\right)^2 - 1$

$y = \pm\sqrt{\left(-\ln x + C\right)^2 - 1}$ explicit solution

6. $dr + r\cot\theta\, d\theta = d\theta$

$$\frac{dr}{d\theta} + r\underbrace{\cot\theta}_{P(\theta)} = \underbrace{1}_{Q(\theta)}$$

$$e^{\int P(\theta)d\theta} = e^{\int \cot\theta d\theta} = e^{-\ln(\csc\theta)} = e^{\ln(\csc\theta)^{-1}} = (\csc\theta)^{-1} = \sin\theta$$

$$\sin\theta\left(\frac{dr}{d\theta} + r\cot\theta\right) = (1)\sin\theta$$

$$\frac{d}{d\theta}(\sin\theta \cdot r) = \sin\theta$$

$$\int d(r\sin\theta) = \int \sin\theta\, d\theta$$

$$\boxed{r\sin\theta = -\cos\theta + C}$$ implicit solution

$$r = \frac{-\cos\theta + C}{\sin\theta} = -\cot\theta + C\csc\theta$$ explicit solution

7. $r\sqrt{1-\theta^2}\,\dfrac{dr}{d\theta} = \theta + 4$ separable

$$r\,dr = \frac{\theta+4}{\sqrt{1-\theta^2}}\,d\theta$$

$$\int r\,dr = \int\left(\frac{\theta}{\sqrt{1-\theta^2}} + \frac{4}{\sqrt{1-\theta^2}}\right)d\theta$$

$u = 1-\theta^2$
$du = -2\theta\,d\theta$
$-\frac{1}{2}du = \theta\,d\theta$

$$\frac{1}{2}r^2 = -\frac{1}{2}\int u^{-1/2}\,d\theta + 4\sin^{-1}\theta$$

$$\frac{1}{2}r^2 = -\frac{1}{2}\cdot\frac{2}{1}u^{1/2} + 4\sin^{-1}\theta + C$$

$$\boxed{\frac{1}{2}r^2 = -(1-\theta^2)^{1/2} + 4\sin^{-1}\theta + C}$$ implicit solution

to find the explicit solution:

$$\frac{1}{2}r^2 = -(1-\theta^2)^{1/2} + 4\sin^{-1}\theta + C_1$$

$$r^2 = -2(1-\theta^2)^{1/2} + 8\sin^{-1}\theta + \boxed{2C_1}$$ $C = 2C_1$

$$r = \pm\sqrt{-2(1-\theta^2)^{1/2} + 8\sin^{-1}\theta + C}$$

8. $2\frac{dy}{dx} = 5-6y$

separable (but linear first-order with $P(x) = -3$ also works)

$$\frac{1}{5-6y}\,dy = \frac{1}{2}\,dx$$

$$\int \frac{1}{-6y+5}\,dy = \int \frac{1}{2}\,dx$$

$$\boxed{\frac{1}{-6}\ln(-6y+5) = \frac{1}{2}x + C}$$

implicit solution

to find the explicit solution: $-\frac{1}{6}\ln(-6y+5) = \frac{1}{2}x + C_1$

$$\ln(-6y+5) = -3x - 6C_1$$

$$-6y+5 = e^{-3x-6C_1}$$

$$-6y = e^{-3x}e^{-6C_1} - 5$$

$$y = e^{-3x}\boxed{\frac{e^{-6C_1}}{-6}} + \frac{5}{6}$$

$$C = \frac{e^{-6C_1}}{-6}$$

$$y = Ce^{-3x} + \frac{5}{6}$$

9. $e^{2x}\,dy + e^{x}\,dx = 4\,dx$

$$e^{2x}\,dy = 4\,dx - e^{x}\,dx$$

$$e^{2x}\,dy = (4 - e^{x})\,dx$$

separable

$$dy = \frac{4 - e^{x}}{e^{2x}}\,dx$$

$$\int dy = \int\left(\frac{4}{e^{2x}} - \frac{e^{x}}{e^{2x}}\right)dx$$

$$y = \int\left(4e^{-2x} - e^{-x}\right)dx$$

$$y = 4\left(\frac{1}{-2}\right)e^{-2x} - \left(\frac{1}{-1}\right)e^{-x} + C$$

$$\boxed{y = -2e^{-2x} + e^{-x} + C}$$

10. $\frac{dv}{dt} - \frac{v}{t} = \ln t$

$$\frac{dv}{dt} \underbrace{-\frac{1}{t}}_{P(t)} v = \underbrace{\ln t}_{Q(t)}$$

$$e^{\int P(t)dt} = e^{\int -\frac{1}{t}dt} = e^{-\ln t} = e^{\ln t^{-1}} = t^{-1}$$

$$t^{-1}\left(\frac{dv}{dt} - \frac{1}{t}v\right) = (\ln t)\, t^{-1}$$

$$\frac{d}{dt}\left(t^{-1}v\right) = \frac{\ln t}{t}$$

$$\int d(t^{-1}v) = \int \frac{\ln t}{t}\, dt$$

$$u = \ln t$$
$$du = \frac{1}{t}dt$$

$$t^{-1}v = \int u\, du$$

$$\frac{v}{t} = \frac{1}{2}u^2 + C$$

$$\boxed{\frac{v}{t} = \frac{1}{2}(\ln t)^2 + C}$$ implicit solution

$$v = \frac{1}{2}t(\ln t)^2 + Ct$$ explicit solution

11. $(yx^2 + y)\dfrac{dy}{dx} = \tan^{-1}x$

$y(x^2+1)\dfrac{dy}{dx} = \tan^{-1}x$ separable

$y\,dy = \dfrac{\tan^{-1}x}{x^2+1}\,dx$

$\int y\,dy = \int \tan^{-1}x \cdot \dfrac{1}{1+x^2}\,dx$

let $u = \tan^{-1}x$

$du = \dfrac{1}{1+x^2}\,dx$

$\frac{1}{2}y^2 = \int u\,du$

$\frac{1}{2}y^2 = \frac{1}{2}u^2 + C$

$\boxed{\frac{1}{2}y^2 = \frac{1}{2}(\tan^{-1}x)^2 + C}$ implicit solution

to find the explicit solution:

$\frac{1}{2}y^2 = \frac{1}{2}(\tan^{-1}x)^2 + C_1$

$y^2 = (\tan^{-1}x)^2 + \boxed{2C_1}$ $C = 2C_1$

$y = \pm\sqrt{(\tan^{-1}x)^2 + C}$ explicit solution

12. $V+1+(1+\sin T)\sec T\,\dfrac{dV}{dT}=0$

$$(1+\sin T)\sec T\,\frac{dV}{dT} = -(V+1)$$ separable

$$\frac{1}{V+1}\,dV = -\frac{1}{(1+\sin T)\sec T}\,dT$$

$$\int \frac{1}{V+1}\,dV = -\int \frac{\cos T}{1+\sin T}\,dT$$

let $u = 1+\sin T$
$du = \cos T\,dT$

$$\ln(V+1) = -\int \frac{1}{u}\,du$$

$$\ln(V+1) = -\ln u + C$$

$$\boxed{\ln(V+1) = -\ln(1+\sin T) + C}$$ implicit solution

to find the explicit solution:

$$\ln(V+1) = -\ln(1+\sin T) + C_1$$

$$V+1 = e^{-\ln(1+\sin T)+C_1}$$

$$V+1 = e^{-\ln(1+\sin T)}\boxed{e^{C_1}}$$ $C = e^{C_1}$

$$V = C e^{\ln(1+\sin T)^{-1}} - 1$$

$$V = C(1+\sin T)^{-1} - 1$$

$$V = \frac{C}{1+\sin T} - 1$$