Section 31.7/8: Complex Roots

Monday, January 8, 2018 10:24 AM

suppose we have a 2^{nd} order linear homogeneous DE that when we solve the auxiliary equation, we get solutions of the form

$$m = a \pm bi \qquad \text{where } a \text{ and } b \text{ are real}$$

what do the solutions to the DE look like?
we'll need the fact that

$$e^{ix} = \cos x + i \sin x$$

now, because $m = a \pm bi$, so we have two distinct roots, then

$$
\begin{aligned}
y &= c_1 e^{m_1 x} + c_2 e^{m_2 x} \\
&= c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x} \\
&= c_1 e^{ax + bix} + c_2 e^{ax - bix} \\
&= c_1 e^{ax} e^{bix} + c_2 e^{ax} e^{-bix} \\
&= c_1 e^{ax} (\cos bx + i \sin bx) \\
&\quad + c_2 e^{ax} (\cos(-bx)) + i \sin(-bx))
\end{aligned}
$$

now recall that $\cos x$ is an even function, while $\sin x$ is odd, so $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$

$$y = c_1 e^{ax}(\cos bx + i\sin bx) + c_2 e^{ax}(\cos bx - i\sin bx)$$

$$= \underbrace{(c_1 + c_2)}_{\text{new constant } A} e^{ax}\cos bx + \underbrace{(c_1 - c_2)i}_{\text{new constant } B} e^{ax}\sin bx$$

$$= A e^{ax}\cos bx + B e^{ax}\sin bx$$

$$= e^{ax}(A\cos bx + B\sin bx)$$