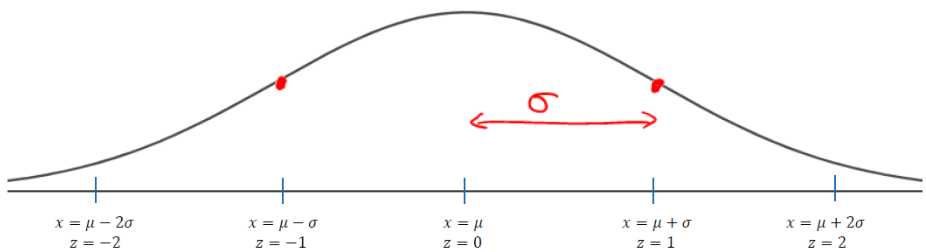
7 The Normal Distribution

The most important and most widely used distribution in statistics is the normal distribution.

Definition: A continuous random variable, X, with mean μ and SD σ has a <u>normal distribution</u> if its p.d.f. is

$$f(x) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}.$$



By use of a change of variable called the <u>z-score</u>,

$$z = \frac{x-\mu}{\sigma},$$

z-score is the number of standard deviations away from the mean that you are at

we can standardize any normal distribution. The <u>standard normal distribution</u> has the p.d.f.

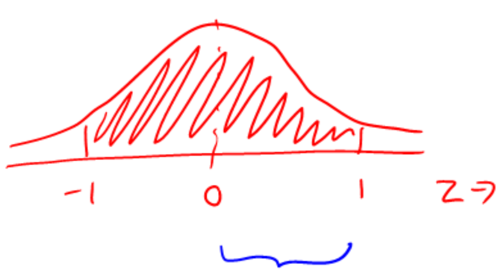
$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}.$$

The graph of a normal distribution is often referred to as a *bell curve.* Since the p.d.f. is very difficult to integrate, we use the *Standard Normal Distribution Table* to find the area under the bell curve corresponding to

$$\int_a^b f(x)\,dx = P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le z \le \frac{b-\mu}{\sigma}\right).$$

Example 7.1. Find the proportion of x-values that are within one standard deviation of the mean for a normal distribution; that is, find $P(\mu - \sigma \le X \le \mu + \sigma)$.

z = 1 and −1



area = 0.3413

$P = 2(0.3413)$

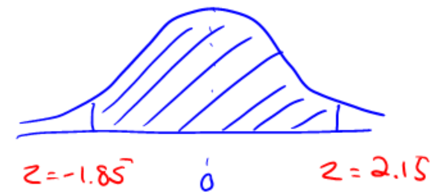
$= 0.6826$

or 68%

Example 7.2. The volume in bottles of gingerale is normally distributed with a mean of 2.01 L and a SD of 0.13 L. Find the probability that a bottle has a volume

(a) between 1.77 and 2.29 L.

$$z_{low} = \frac{x_{low} - \mu}{\sigma} = \frac{1.77 - 2.01}{0.13} = -1.85$$



area = 0.4842

area = 0.4678

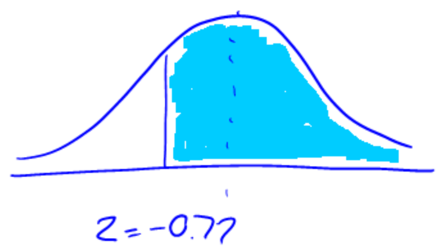
$$z_{high} = \frac{x_{high} - \mu}{\sigma} = \frac{2.29 - 2.01}{0.13} = 2.15$$

$$P(-1.85 < z < 2.15) = 0.4842 + 0.4678$$
$$= 0.952$$

or $\boxed{95.2\%}$

(b) more than 1.91 L.

$$z = \frac{x - \mu}{\sigma} = \frac{1.91 - 2.01}{0.13} = -0.77$$



area = 0.5

area 0.2794

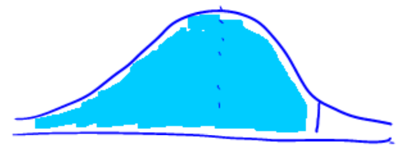
$$P(z > -0.77) \boxed{= 0.7794 \\ = 78\%}$$

Example 7.3. [1, p. 45] Mopeds are popular in Europe because of their mobility, ease of operation, and low cost. A 2008 article described a rolling bench test for determining maximum vehicle speed. A normal distribution with $\mu = 46.8$ km/h and $\sigma = 1.75$ km/h is proposed.

(a) What proportion of mopeds have a maximum speed that is at most 50 km/h?

$$Z_{high} = \frac{x_{high} - \mu}{\sigma} = \frac{50 - 46.8}{1.75} = 1.83$$

$$Z_{low} = \frac{x_{low} - \mu}{\sigma} = \frac{0 - 46.8}{1.75} = -26.74$$



−26.74 is way below

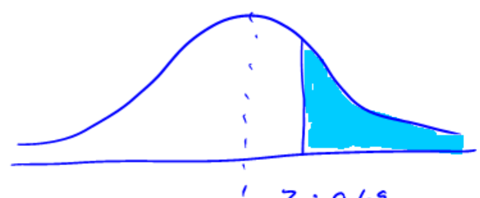
0.5

area 0.4664

$P = 0.9664$ or 97%

(b) What proportion of mopeds have a maximum speed that is at least 48 km/h?

$$Z = \frac{x - \mu}{\sigma} = \frac{48 - 46.8}{1.75} = 0.69$$

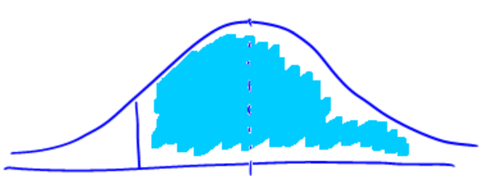


area = 0.2549

$P(z > 0.69) = 0.5 - 0.2549$

$\boxed{= 0.2451}$

(c) What speed separates the fastest 75% of all mopeds from the others?



area = 0.25

$z = -0.675$

$$z = \frac{x - \mu}{\sigma}$$

$x = \mu + z\sigma$

$= 46.8 + (-0.675)(1.75)$

$= 45.6$ km/h

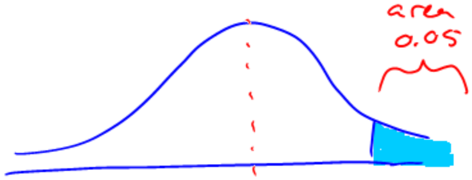
z	0.07	0.08
0.6	0.2486	0.2517

either take the closest value, so $z = 0.67$

or halfway so $z = 0.675$

Example 7.4. [1, p. 45] Suppose the flow of current (in milliamps) in wire strips of a certain type under specified conditions can be modeled with a normal distribution having $\mu = 20$ and $\sigma = 1$. How large must a current flow be to be among the largest 5% of all flows?

area 0.05



z

area 0.45

so $z = 1.645$

$$z = \frac{x - \mu}{\sigma}$$

$$\begin{aligned} x &\geq \mu + z\sigma \\ &\geq 20 + 1.645(1) \\ &\geq 21.645 \\ &\geq 21.6 \text{ mA} \end{aligned}$$

Example 7.5. [1, p. 41] The time that it takes a driver to react to the brake on a decelerating vehicle is critical in avoiding rear-end collisions. A 1993 article from *Ergonomics* suggests that reaction time for an in-traffic response to a brake signal from standard brake lights can be modeled with a normal distribution having parameters $\mu = 1.25$ sec and $\sigma = 0.46$ sec. In the long run, what proportion of reaction times will

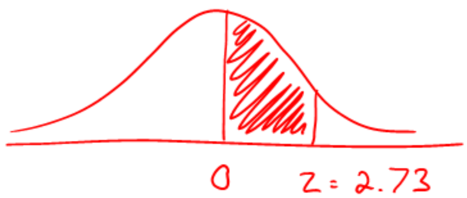
(a) be between 1.00 sec and 1.75 sec?

(b) exceed 2 sec?

Example 7.6. The time it takes to inspect a certain type of production component is normally distributed with a mean of 6.8 s. Find the SD of the inspection times if 26.62% of inspection times are between 6.2 and 7.4 s.

Additional Notes

reading the Standard Normal Table:



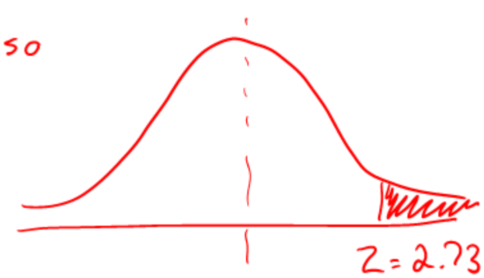
area
= 0.4968

z	0.03
2.7	0.4968

and therefore

area: 2(0.4968) = 0.9936

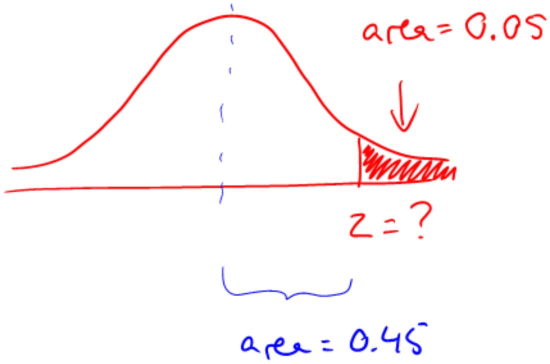
also



area 0.4968

area = 0.5 − 0.4968 = 0.0032

Additional Notes



this is a "reverse look-up"

$z = 1.645$

	0.04	0.05
1.6	0.4495	0.4505

give either the left
or the right
or midway