Section 31.4: cont'd

Tuesday, February 7, 2017 2:01 PM

linear first-order:

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

must both be <u>linear</u> (exponent 1)

example: what are $P(x)$ and $Q(x)$ for:

$$x\,dy - 2x^3 y\,dx = 3x\,dx$$

divide by $x\,dx$

$$\frac{dy}{dx} - 2x^2 y = 3$$

$P(x) = -2x^2$

$Q(x) = 3$

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recall:

$$\frac{d}{dx}(ye^x) = \frac{dy}{dx}e^x + ye^x$$

$$\frac{d}{dx}(yx^2) = \frac{dy}{dx}x^2 + 2yx$$

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how do you solve this type of DE?

- follow the procedure in the handout

example: solve

$$dy - 3y\,dx = e^{3x}\,dx$$

$$\frac{dy}{dx} - 3y = e^{3x}$$

$$\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)}$$

linear first order with

$$P(x) = -3$$
$$Q(x) = e^{3x}$$

calculate the integrating factor

$$IF = e^{\int P(x)\,dx} = e^{\int -3\,dx}$$
$$= e^{-3x}$$

note: omit the $+C$

then, multiply both sides of the DE by the integrating factor:

$$e^{-3x}\left(\frac{dy}{dx} - 3y\right) = \left(e^{3x}\right)e^{-3x}$$

$$e^{-3x}\frac{dy}{dx} - 3ye^{-3x} = 1$$

$$\frac{d}{dx}\underbrace{\left(y\,e^{-3x}\right)}_{y\,\cdot\,\text{integrating factor}} = 1$$

so, rewrite in differential form:

$$\int d\left(ye^{-3x}\right) = \int dx$$

implicit → $ye^{-3x} = x + C$

explicit → $y = e^{3x}(x+c)$

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solve the following, giving an explicit solution.

$$dy + y\,dx = e^{-x}\cos x\,dx$$

$$\frac{dy}{dx} + y = e^{-x}\cos x$$

linear with $P(x) = 1$

$$IF = e^{\int P(x)dx} = e^{\int dx} = e^{x}$$

$$\frac{dy}{dx}e^{x} + ye^{x} = e^{x}e^{-x}\cos x$$

$$\frac{d}{dx}\left(ye^{x}\right) = \cos x$$

$$\int d\left(ye^{x}\right) = \int \cos x\,dx$$

$$ye^{x} = \sin x + C$$

$$y = e^{-x}(\sin x + c)$$

$$\text{or} \quad \frac{\sin x + C}{e^x}$$

Solve the following DE, giving an explicit solution, considering that $y(0) = 1$.

$$\frac{dy}{dx} + 2xy = x$$

linear with $P(x) = 2x$

$$e^{\int P(x)\,dx} = e^{\int 2x\,dx} = e^{x^2}$$

$$\frac{dy}{dx} e^{x^2} + 2xy\,e^{x^2} = x e^{x^2}$$

$$\frac{d}{dx}\left(y e^{x^2}\right) = x e^{x^2}$$

$$\int d\left(y e^{x^2}\right) = \int x e^{x^2}\,dx$$

$$y e^{x^2} = \int e^u \frac{du}{2}$$

let $u = x^2$
$du = 2x\,dx$
$\frac{du}{2} = x\,dx$

$$= \frac{1}{2} e^u + C$$

implicit general $\rightarrow$ $$= \frac{1}{2} e^{x^2} + C$$

explicit general $\rightarrow$ $y = \frac{1}{2} + Ce^{-x^2}$

now $y(0) = 1$ (when $x = 0$, $y = 1$)

$1 = \frac{1}{2} + Ce^{0}$

$C = \frac{1}{2}$

$$\boxed{y = \frac{1}{2} + \frac{1}{2}e^{-x^2}}$$

now, check your answer!

$$y = \frac{1}{2} + \frac{1}{2}e^{-x^2}$$

$$\frac{dy}{dx} = \frac{1}{2}e^{-x^2}(-2x)$$

$$= -xe^{-x^2}$$

plug into DE:

$$\frac{dy}{dx} + 2xy = x$$

$$-xe^{-x^2} + 2x\left(\frac{1}{2} + \frac{1}{2}e^{-x^2}\right) = x$$

$$\cancel{-xe^{-x^2}} + x + \cancel{xe^{-x^2}} = x$$

$$x = x$$

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