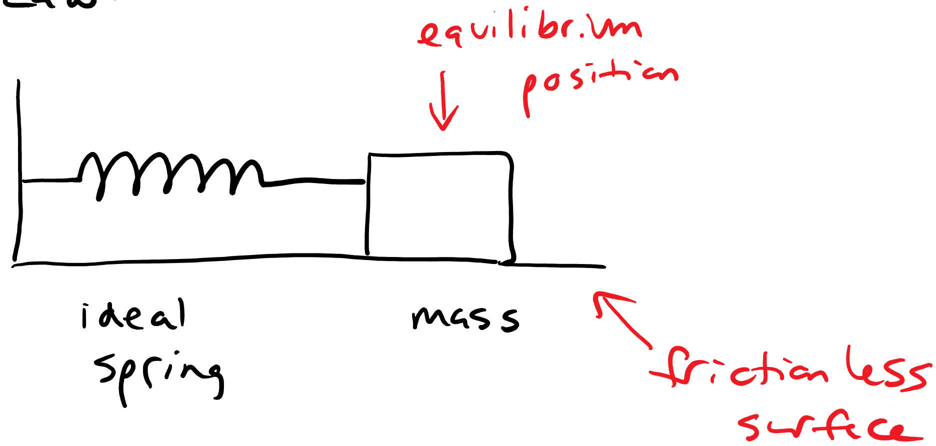
Section 31.10: Applications of Higher-Order Equations

Monday, March 6, 2017 1:04 PM

Hooke's Law:



pull spring to the right by an external force F_{ext}

spring pulls back, $\vec{F}_{spring}$ is to left

and you reach equilibrium when

$$\vec{F}_{spring} = -\vec{F}_{ext}$$

but Hooke's Law says

$$\vec{F}_{spring} = -k\vec{x}$$

if you stretch spring to right, force from spring is to left now let go of block so $F_{ext} = 0$

so spring force is an unbalanced force

and

$$\sum \vec{F} = m\vec{a}$$

$$-kx = m\frac{d^2x}{dt^2}$$ $\leftarrow$ 2nd order linear DE

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rearranging:

$$m\frac{d^2x}{dt^2} + kx = 0$$

(or, if you insist: $m\ddot{x} + kx = 0$)

auxiliary equation (but using n instead of m)

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$n^2 + \frac{k}{m} = 0$$

$$n^2 = -\frac{k}{m}$$

$$n = \pm \sqrt{\frac{k}{m}}\, i$$

$$n = a \pm bi$$

$$x = e^{at}(c_1 \cos bt + c_2 \sin bt)$$

$$x = c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

frequently written as

$$x = c_1 \cos w_0 t + c_2 \sin w_0 t$$

where $w_0 = \sqrt{\frac{k}{m}}$ and is called the natural frequency of oscillation

$\Rightarrow$ simple harmonic motion

example: A marble rolling back and forth in the bottom of a circular bowl follows a path given by:

$$\frac{d^2 s}{dt^2} + 4s = 0$$

where s is the arclength in cm and t is the time in seconds

a) Find the general solution to the DE.

b) If the marble is displaced by 2.5 cm and then let go from rest, what is the arclength s as a function of time?

c) What if, instead, the marble is started at the bottom of the bowl and given a push such that its initial speed is 6 cm/s? What is $s(t)$ then?

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a) aux eqn: $m^2 + 4 = 0$

$$m = \pm 2i$$

$$s = e^{at}(c_1 \cos bt + c_2 \sin bt)$$

$$\boxed{s = c_1 \cos 2t + c_2 \sin 2t}$$

b) $s = 2.5$ cm

$\frac{ds}{dt} = 0$ (starting from rest)

at $t=0$ $\quad s = 2.5 = c_1 \overset{1}{\cancel{\cos(2\cdot 0)}} + c_2 \overset{0}{\cancel{\sin 2(0)}}$

$$2.5 = c_1$$

$$c_1 = 2.5$$

$$\frac{ds}{dt} = -2c_1 \underset{0}{\sin 2t} + 2c_2 \underset{1}{\cos 2t}$$

$$\frac{ds}{dt} = -2c_1 \sin 2t + 2c_2 \cos 2t$$

at $t=0$ $\quad 0 = -2c_1 \cancel{\sin(2\cdot 0)}^{0} + 2c_2 \cancel{\cos(0)}^{1}$

$$0 = 2c_2$$

$$c_2 = 0$$

$$\boxed{s = 2.5 \cos 2t}$$

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c) at $t=0$, $s=0$ and $\frac{ds}{dt} = 6$ cm/s

$$s = c_1 \cos 2t + c_2 \sin 2t$$

$t=0$ $\quad 0 = c_1 \cdot 1 + c_2 \cdot 0$

$$c_1 = 0$$

so $s = c_2 \sin 2t$

$$\frac{ds}{dt} = 2c_2 \cos 2t$$

at $t=0$, $\quad \frac{ds}{dt} = 6 = 2c_2 \cos \cancel{2\cdot 0}^{1}$

$$c_2 = 3$$

$$\boxed{s = 3 \sin 2t}$$