Practice Questions #1-15

1. $$\int_0^1 \frac{e^{9x}}{2+4e^{9x}}\,dx$$

$u = 2+4e^{9x}$

$du = 36e^{9x}\,dx$

$\frac{1}{36}\,du = e^{9x}\,dx$

$$= \frac{1}{36}\int_{x=0}^{x=1} \frac{1}{u}\,du$$

$$= \frac{1}{36}\ln|u|\Big|_{x=0}^{x=1}$$

$$= \frac{1}{36}\ln\left|2+4e^{9x}\right|\Big|_0^1$$

Note: since $2+4e^{9x}>0$ for all x, we can drop the absolute value signs

$$= \frac{1}{36}\left[\ln\left(2+4e^{9(1)}\right) - \ln\left(2+4e^{9(0)}\right)\right]$$

$$= \frac{1}{36}\left[\ln\left(2+4e^{9}\right) - \ln 6\right]$$

$$= \frac{1}{36}\ln\left(\frac{2+4e^9}{6}\right) = \frac{1}{36}\ln\left(\frac{1+2e^9}{3}\right) \approx 0.24$$

2. $$\int_0^{\pi/4} \sqrt[4]{\tan x}\,\sec^2 x\,dx$$

$u = \tan x$

$du = \sec^2 x\,dx$

$$= \int_0^{\pi/4} (\tan x)^{1/4}\sec^2 x\,dx$$

$$= \int_{x=0}^{x=\pi/4} u^{1/4}\,du$$

$$= \frac{4}{5}u^{5/4}\Big|_{x=0}^{x=\pi/4}$$

$$= \frac{4}{5}(\tan x)^{5/4}\Big|_0^{\pi/4}$$

$$= \frac{4}{5}\left[\left(\tan\frac{\pi}{4}\right)^{5/4} - (\tan 0)^{5/4}\right]$$

$$= \frac{4}{5}\left[1^{5/4} - 0^{5/4}\right] = \frac{4}{5}$$

3. $\int \frac{x}{\sqrt{4-x^2}}\,dx$

$= \int x\,(4-x^2)^{-1/2}\,dx$

$u = 4 - x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$= -\frac{1}{2}\int u^{-1/2}\,du$

$= -\frac{1}{2}\cdot\frac{2}{1}u^{1/2} + C$

$= -\sqrt{4-x^2} + C$

4. $\int \frac{4}{\sec x\, e^{\sin x}}\,dx$

$= 4\int \cos x\, e^{-\sin x}\,dx$

$u = -\sin x$

$du = -\cos x\,dx$

$-du = \cos x\,dx$

$= -4\int e^u\,du$

$= -4e^u + C$

$= -4e^{-\sin x} + C$

$= \frac{-4}{e^{\sin x}} + C$

5. $\displaystyle\int \frac{x}{\sqrt{1+x}}\,dx$

$$= \int x(1+x)^{-1/2}\,dx$$

METHOD 1: integration by parts

D	I
x	$(1+x)^{-1/2}$
1	$2(1+x)^{1/2}$

$$= x \cdot 2(1+x)^{1/2} - \int 2(1+x)^{1/2}\,dx$$

$$= 2x(1+x)^{1/2} - 2 \cdot \tfrac{2}{3}(1+x)^{3/2} + C$$

$$= 2x(1+x)^{1/2} - \tfrac{4}{3}(1+x)^{3/2} + C$$

METHOD 2: substitution

$u = 1+x \quad \Rightarrow \quad x = u-1$

$du = dx$

$$\int x(1+x)^{-1/2}\,dx$$

$$= \int (u-1)\,u^{-1/2}\,du$$

$$= \int \left(u^{1/2} - u^{-1/2}\right) du$$

$$= \tfrac{2}{3}u^{3/2} - 2u^{1/2} + C$$

$$= \tfrac{2}{3}(1+x)^{3/2} - 2(1+x)^{1/2} + C$$

Note: these answers look different but are the same: both factor to $2(1+x)^{1/2}\left(\frac{1}{3}x - \frac{2}{3}\right) + C$

6. $$\int \frac{e^{-x}}{1+e^{-2x}}\,dx$$

$$= \int \frac{e^{-x}}{1+\left(e^{-x}\right)^2}\,dx$$

$u = e^{-x}$
$du = -e^{-x}\,dx$
$-du = e^{-x}\,dx$

$$= -\int \frac{1}{1+u^2}\,du$$

$$= -\tan^{-1} u + C$$

$$= -\tan^{-1}\left(e^{-x}\right) + C$$

7. $$\int_0^{\pi/4} \frac{5 - 8\sin(2x)}{\cos^2 x}\,dx$$

$$= \int_0^{\pi/4} \left(\frac{5}{\cos^2 x} - \frac{8 \cdot 2\sin x \cancel{\cos x}}{\cos^{\cancel{2}} x}\right) dx$$

$\sin 2x = 2\sin x\cos x$

$$= \int_0^{\pi/4} \left(5\sec^2 x - 16\tan x\right) dx$$

$$= 5\tan x - 16\ln|\sec x| \Big|_0^{\pi/4}$$

$$= \left(5\tan\tfrac{\pi}{4} - 16\ln\left|\sec\tfrac{\pi}{4}\right|\right) - \left(5\tan 0 - 16\ln|\sec 0|\right)$$

$$= 5\cdot 1 - 16\ln\sqrt{2} - 5\cdot 0 + 16\cdot 0$$

$$= 5 - 16\ln\sqrt{2}$$

$$= 5 - 8\ln 2 \approx -0.55$$

8. $\int \frac{\ln x}{x^2}\,dx$

$= \int x^{-2} \ln x \, dx$

D	I
$\ln x$	x^{-2}
$\frac{1}{x}$	$-x^{-1}$

$+$ $-$

$= (\ln x)(-x^{-1}) - \int -x^{-1} \frac{1}{x}\,dx$

$= -\frac{\ln x}{x} + \int x^{-2}\,dx$

$= -\frac{\ln x}{x} + (-1)x^{-1} + C$

$= -\frac{\ln x}{x} - \frac{1}{x} + C$

9. $\int \frac{3}{x^2-25}\,dx$

$\frac{3}{(x+5)(x-5)} = \frac{A}{x+5} + \frac{B}{x-5}$

$3 = A(x-5) + B(x+5)$

$x=5 \Rightarrow \quad 3 = 0 + B(10)$

$B = \frac{3}{10}$

$x=-5 \Rightarrow \quad 3 = A(-10) + 0$

$A = -\frac{3}{10}$

$\int \frac{3}{x^2-25}\,dx = \int \left(\frac{-\frac{3}{10}}{x+5} + \frac{\frac{3}{10}}{x-5} \right) dx$

$= -\frac{3}{10} \ln|x+5| + \frac{3}{10} \ln|x-5| + C \quad = -\frac{3}{10} \ln\left|\frac{x+5}{x-5}\right| + C$

10. $$\int \frac{5}{x^2+8x+17}\,dx$$

$$x^2+8x+17 = x^2+8x+16-16+17$$
$$= (x+4)^2+1$$

$$= \int \frac{5}{(x+4)^2+1}\,dx$$

$$= 5\tan^{-1}(x+4) + C$$

11. $$f(x,y) = e^x \cos y + e^{-2x}\tan y$$

$$\frac{\partial f}{\partial x} = e^x \cos y - 2e^{-2x}\tan y$$

$$\frac{\partial f}{\partial y} = -e^x \sin y + e^{-2x}\sec^2 y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= -e^x \sin y - 2e^{-2x}\sec^2 y$$

12. $\int_1^2 \int_x^{x^2} x^2 y \, dy \, dx$

$$= \int_1^2 \frac{1}{2} x^2 y^2 \Big|_{y=x}^{y=x^2} dx$$

$$= \frac{1}{2} \int_1^2 \left[x^2 (x^2)^2 - x^2 x^2 \right] dx$$

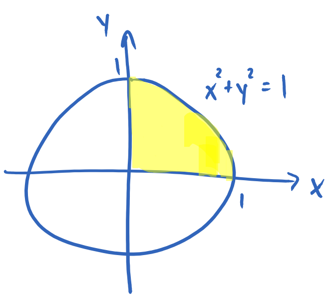
$$= \frac{1}{2} \int_1^2 (x^6 - x^4) \, dx$$

$$= \frac{1}{2} \left(\frac{1}{7} x^7 - \frac{1}{5} x^5 \right) \Big|_1^2$$

$$= \frac{1}{2} \left[\left(\frac{1}{7} \cdot 2^7 - \frac{1}{5} \cdot 2^5 \right) - \left(\frac{1}{7} \cdot 1^7 - \frac{1}{5} \cdot 1^5 \right) \right]$$

$$= \frac{209}{35} \approx 5.97$$

13.



polar coordinates:

$z = xy = r\cos\theta \, r\sin\theta = r^2 \cos\theta \sin\theta$

$0 \le r \le 1$

$0 \le \theta \le \frac{\pi}{2}$

$$V = \iint z \, dA = \int_0^{\pi/2} \int_0^1 r^2 \cos\theta \sin\theta \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^1 r^3 \cos\theta \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4} r^4 \cos\theta \sin\theta \Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4} \cos\theta \sin\theta \left(1^4 - 0^4 \right) d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4} \cos\theta \sin\theta \, d\theta$$

$u = \sin\theta$
$du = \cos\theta \, d\theta$

$$= \int_{\theta=0}^{\theta=\pi/2} \frac{1}{4} u \, du$$

$$= \frac{1}{8} u^2 \Big|_{\theta=0}^{\theta=\pi/2}$$

$$= \frac{1}{8} \sin^2\theta \Big|_0^{\pi/2}$$

$$= \frac{1}{8} \left(\sin^2 \frac{\pi}{2} - \sin^2 0 \right)$$

$$= \frac{1}{8} \left(1^2 - 0^2 \right)$$

$$= \frac{1}{8}$$

14. $y = \sqrt{Cx - x^2} = (Cx - x^2)^{1/2} \Rightarrow y' = \frac{1}{2}(Cx - x^2)^{-1/2}(C - 2x)$

$$y' = \frac{C - 2x}{2\sqrt{Cx - x^2}}$$

so $2xyy' + x^2 = y^2$

$$\cancel{2}x \cancel{\sqrt{Cx - x^2}} \cdot \frac{C - 2x}{\cancel{2} \cancel{\sqrt{Cx - x^2}}} + x^2 = \left(\sqrt{Cx - x^2} \right)^2$$

$$Cx - 2x^2 + x^2 = Cx - x^2$$

$$Cx - x^2 = Cx - x^2$$

15. $\sin x \, dy = x^3 \sin^2 x \, dx + \frac{3y \sin x}{x} dx$

$$\sin x \frac{dy}{dx} = x^3 \sin^2 x + \frac{3y\sin x}{x}$$

$$\sin x \frac{dy}{dx} - \frac{3 \sin x}{x} y = x^3 \sin^2 x$$

$$\frac{dy}{dx} \underbrace{- \frac{3}{x}}_{P(x)} y = x^3 \sin x$$

linear first-order

$$e^{\int P(x)\,dx} = e^{\int -\frac{3}{x} dx}$$
$$= e^{-3 \ln x}$$
$$= e^{\ln x^{-3}}$$
$$= x^{-3}$$

$$x^{-3} \frac{dy}{dx} - x^{-3} \frac{3}{x} y = x^{-3} x^3 \sin x$$

$$x^{-3} \frac{dy}{dx} - 3x^{-4} y = \sin x$$

$$\frac{d}{dx}\left(x^{-3} y\right) = \sin x$$

$$\int d\left(x^{-3} y\right) = \int \sin x \, dx$$

$$x^{-3} y = -\cos x + C$$

implicit solution $\Rightarrow$ don't need to solve for y

$y = 1$ when $x = \pi$

$$\pi^{-3}(1) = -\cos \pi + C$$
$$\frac{1}{\pi^3} = 1 + C$$
$$C = \frac{1}{\pi^3} - 1$$

so $\quad x^{-3} y = -\cos x + \frac{1}{\pi^3} - 1$