Math 193 - Practice for Test 2 Solutions

1. Consider the solution $y = c^2 + \frac{c}{x}$ to the differential equation $x^4 (y')^2 - xy' = y$.

(4 points)

a) State the order of the DE. first order, 1

b) Is this solution a general or particular solution? general 1

c) Show that the given solution really is a solution to the DE.

$y = c^2 + \frac{c}{x}$

$y' = -\frac{c}{x^2}$ 1

sub into DE. $x^4 (y')^2 - xy' = y$

$x^4 \left(-\frac{c}{x^2}\right)^2 - x\left(-\frac{c}{x^2}\right) = c^2 + \frac{c}{x}$

$x^4 \frac{c^2}{x^4} + \frac{xc}{x^2} = c^2 + \frac{c}{x}$ 1

$c^2 + \frac{c}{x} = c^2 + \frac{c}{x}$ ✓

2. Solve the following DE. You may leave your answer in implicit form.

$$\sqrt{1+4x^2}\ dy = y^3 x dx$$

$$\frac{dy}{y^3} = \frac{x\,dx}{\sqrt{1+4x^2}}$$ separable

(3)

let $u = 1+4x^2$

$du = 8x\,dx$

$$\frac{dy}{y^3} = \frac{1}{8}\frac{du}{\sqrt{u}}$$

$$\int y^{-3}\,dy = \int \frac{1}{8} u^{-1/2}\,du$$

$$\frac{y^{-2}}{2} = \frac{2}{8} u^{1/2} + C$$

$$\boxed{-\frac{1}{2y^2} = \frac{1}{4}\sqrt{1+4x^2} + C}$$

or

$$\boxed{\frac{-2}{y^2} = \sqrt{1+4x^2} + C}$$

or

$$\boxed{\sqrt{1+4x^2} + \frac{2}{y^2} = C}$$

etc

3. Solve the <u>linear</u> differential equation, subject to the given initial condition. Give an explicit solution. (6 points)

$$3x\,dy - y\,dx = 9x\,dx$$

where $y(1) = 3$

$$\frac{dy}{dx} - \frac{1}{3x}\,y = 3$$

1

linear with $P(x) = -\frac{1}{3x}$ and integrating factor

$$e^{\int P(x)\,dx} = e^{\int -\frac{1}{3x}\,dx} \quad 1$$

$$= e^{-\frac{1}{3}\ln x} = e^{\ln x^{-1/3}} = x^{-1/3} \quad 1$$

$$x^{-1/3}\frac{dy}{dx} - \frac{x^{-4/3}}{3}\,y = 3x^{-1/3} \quad 1$$

$$\frac{d}{dx}\left(x^{-1/3}y\right) = 3x^{-1/3}$$

$$\int d\left(x^{-1/3}y\right) = \int 3x^{-1/3}\,dx$$

$$x^{-1/3}y = \frac{3x^{2/3}}{2/3} + C$$

$$y = \tfrac{9}{2}x + Cx^{1/3} \quad 1$$

when $x = 1,\ y = 3$:

$$3 = \tfrac{9}{2}\cdot 1 + C$$

$$C = 3 - \tfrac{9}{2} = -\tfrac{3}{2}$$

so $\boxed{y = \tfrac{9}{2}x - \tfrac{3}{2}x^{1/3}}$ 1

4.

$$(xy+y)y' = 2$$

$$(xy+y)\frac{dy}{dx} = 2$$

$$y(x+1)\frac{dy}{dx} = 2$$

$$\int y\,dy = \int \frac{2\,dx}{x+1}$$

$$\frac{y^2}{2} = 2\ln|x+1| + C$$

but at $x=0$, $y=2$:

$$\frac{4}{2} = 2\ln 1 + C$$

$$C = 2$$

so $\boxed{\frac{y^2}{2} = 2\ln|x+1| + 2}$

or $\boxed{y^2 = 4\ln|x+1| + 4}$

5. Solve the following linear differential equation under the given conditions. Give an explicit answer.

$$\frac{dv}{dt} - \frac{v}{t} = \ln t \qquad \text{given that when } t = 1,\ v = 8.$$

linear with integrating factor

$$e^{\int P(t)dt} = e^{\int -\frac{1}{t}dt}$$

$$= e^{-\ln t} = e^{\ln 1/t} = 1/t$$

1

then multiply both sides:

$$\frac{dv}{dt}\cdot\frac{1}{t} - \frac{v}{t^2} = \frac{\ln t}{t}$$

$$\frac{d}{dt}\left(v\cdot\frac{1}{t}\right) = \frac{\ln t}{t}$$

5

$$v\cdot\frac{1}{t} = \int \frac{\ln t}{t}\,dt \qquad 2$$

let $u = \ln t$

$du = \frac{1}{t}dt$

$$= \int u\,du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{(\ln t)^2}{2} + C$$

so $v = \frac{t(\ln t)^2}{2} + Ct$ 1

$$\boxed{v = \frac{t(\ln t)^2}{2} + 8t}$$

1

but $v(1) = 8$

$8 = 0 + C$ (since $\ln 1 = 0$)

$C = 8$

6. In his latest attempt to catch the roadrunner, the wily coyote has ordered some rocket-powered rollerskates from ACME Corp. To test the skates out, he straps them on and, starting from rest, fires the rockets. After ignition, the coyote finds that the acceleration of the skates (and the unhappy coyote) towards the nearest cliff is proportional to the square root of the velocity, and that after exactly 3 seconds he has reached a velocity of 144 m/s. (6 points)

a) Write an equation showing the relationship between the coyote's acceleration and his velocity. Then change your equation into a DE using the fact that the acceleration is the rate of change of the velocity.
b) Solve the DE in part (a) to find an expression for the coyote's velocity as a function of the time elapsed after rocket ignition.
c) At what time is the coyote zooming towards the cliff at 100 m/s?

a) $a \propto \sqrt{v}$ so $a = k\sqrt{v}$

$$\boxed{\frac{dv}{dt} = k\sqrt{v}}$$ 1

b) $$\frac{dv}{\sqrt{v}} = k\,dt$$ separable 1

$$\int v^{-1/2}\,dv = \int k\,dt$$

$$2\sqrt{v} = kt + C$$ 1

at $t=0, v=0$ (starting from rest), so $0 = 0 + C$
$C = 0$ 1

at $t=3, v=144$, so $2\cdot\sqrt{144} = k\cdot 3$
$24 = 3k$
$k = 8$

and $2\sqrt{v} = 8t$
$\sqrt{v} = 4t$
$$\boxed{v = 16t^2}$$ 1

c) $v = 100$, so
$100 = 16t^2$
$t^2 = \frac{100}{16} = \frac{25}{4}$

$$\boxed{t = 2.5\text{ s}}$$ 1

7

auxiliary eqn: $m^2 - 8m + k = 0$

quadratic: $m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{8 \pm \sqrt{64 - 4k}}{2}$

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a) $k = 17$ so $m = \dfrac{8 \pm \sqrt{-4}}{2} = 4 \pm i = \alpha \pm \beta i$ for $\alpha = 4$, $\beta = 1$ (complex roots)

$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

$\boxed{= e^{4x}(C_1 \cos x + C_2 \sin x)}$

b) $k = 16$ so $m = \frac{8}{2} = 4$ repeated real root

$y = (C_1 + C_2 x)\, e^{mx}$

$\boxed{= (C_1 + C_2 x)\, e^{4x}}$

c) $k = 15$ so $m = \dfrac{8 \pm \sqrt{4}}{2} = 4 \pm 1 = 5, 3$ two distinct real roots

$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$\boxed{= C_1 e^{3x} + C_2 e^{5x}}$

d) $k = 14$ so $m = \dfrac{8 \pm \sqrt{8}}{2} = 4 \pm \sqrt{2}$ two distinct real

$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$\boxed{= C_1 e^{(4+\sqrt{2})x} + C_2 e^{(4-\sqrt{2})x}}$

(8) State the form of the particular solution y_p for the following. Leave your answer with undetermined coefficients. (This means "Write down your initial guess for y_p but don't bother to solve for the constants.") Please note that the complementary solution for the homogeneous equation is $y_c = C_1 e^{2x} + C_2 e^{3x}$.

a) $y'' - 5y' + 6y = 4x^2 - 3$ $\quad y_p = Ax^2 + Bx + C$

b) $y'' - 5y' + 6y = x^2 e^{-7x}$ $\quad y_p = Ax^2 e^{-7x} + Bxe^{-7x} + Ce^{-7x}$

c) $y'' - 5y' + 6y = e^x \sin x$ $\quad y_p = Ae^x \sin x + Be^x \cos x$

d) $y'' - 5y' + 6y = 4e^{3x}$ $\quad y_p = \underbrace{Axe^{3x}}$

need extra x because this is the "bad case" - y_c already contains an e^{3x} term

(9) Solve $y'' + 4y' = -4(y + 2x)$.

$y'' + 4y' + 4y = -8x$

y_c: $m^2 + 4m + 4 = 0$

$m = -2$ (repeated)

so $\boxed{y_c = (C_1 + C_2 x)e^{-2x}}$

y_p: $y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

$y'' + 4y' + 4y = -8x$

$0 + 4A + 4Ax + 4B = -8x$

$\underbrace{4Ax}_{\text{so this equals } -8x} + \underbrace{(4A + 4B)}_{\text{this equals zero}} = -8x$

$4A = -8$

$A = -2$

$4A + 4B = 0$

$B = -A = 2$

$\therefore \boxed{y_p = -2x + 2}$

(3)

$$\boxed{y = y_c + y_p = (C_1 + C_2 x)e^{-2x} - 2x + 2}$$

10 Solve: $y'' + 2y' - 3y = 2 + 12e^{-3x}$.

complementary soln:

$$m^2 + 2m - 3 = 0$$

$$(m+3)(m-1) = 0$$

$$m = 1, -3$$

$$y_c = c_1 e^{x} + c_2 e^{-3x} \quad (1)$$

particular solution:

$$y_p = A + \cancel{Be^{-3x}} \quad \text{bad case}$$

bad case omitted so B's cancel: (-2)

$$= A + Bxe^{-3x} \quad (1)$$

$$y_p' = Be^{-3x} - 3Bxe^{-3x}$$

$$y_p'' = -3Be^{-3x} - 3Be^{-3x} + 9Bxe^{-3x}$$

$$= -6Be^{-3x} + 9Bxe^{-3x} \quad (1)$$

sub back into DE:

$$y'' + 2y' - 3y = 2 + 12e^{-3x}$$

$$-6Be^{-3x} + 9Bxe^{-3x} + 2(Be^{-3x} - 3Bxe^{-3x}) - 3(A + Bxe^{3x}) = 2 + 12e^{-3x}$$

$$-6Be^{-3x} + \cancel{9Bxe^{-3x}} + 2Be^{-3x} - \cancel{6Bxe^{-3x}} - 3A - \cancel{3Bxe^{-3x}} = 2 + 12e^{-3x}$$

$$-4Be^{-3x} - 3A = 2 + 12e^{-3x} \quad (1)$$

$$-3A = 2 \qquad \text{and} \qquad -4B = 12$$

$$A = -2/3 \qquad\qquad B = -3$$

so $y_p = -2/3 - 3xe^{-3x}$ (1)

and $\boxed{y = y_c + y_p = c_1 e^{x} + c_2 e^{-3x} - 2/3 - 3xe^{-3x}}$

11 Solve the following differential equation.

$$y'' + 4y' - 21y = 5e^{2x} \qquad \text{if when } x = 0,\ y = -\frac{5}{9} \text{ and } y' = -\frac{1}{9}$$

4

y_c:

$m^2 + 4m - 21 = 0$

$(m+7)(m-3) = 0$

$m = 3, -7$

$$\boxed{y_c = C_1 e^{3x} + C_2 e^{-7x}}$$

y_p:

$y_p = Ae^{2x}$

$y_p' = 2Ae^{2x}$

$y_p'' = 4Ae^{2x}$

← note: not "like" with any term in y_c so not "bad case"

$y'' + 4y' - 21y = 5e^{2x}$

$4Ae^{2x} + 8Ae^{2x} - 21Ae^{2x} = 5e^{2x}$

$(4A + 8A - 21A)e^{2x} = 5e^{2x}$

$-9A = 5$

$A = -5/9$

$$\therefore \boxed{y_p = -5/9\, e^{2x}}$$

$$\boxed{y = y_c + y_p = C_1 e^{3x} + C_2 e^{-7x} - 5/9\, e^{2x}}$$

$y' = 3C_1 e^{3x} - 7C_2 e^{-7x} - 10/9\, e^{2x}$

at $x = 0$,

$y = -5/9 = C_1 + C_2 - 5/9$

$0 = C_1 + C_2$

$C_1 = -C_2$

at $x = 0$,

$y' = -1/9 = 3C_1 - 7C_2 - 10/9$

$1 = 3C_1 - 7C_2$

$1 = 3C_1 + 7C_1$

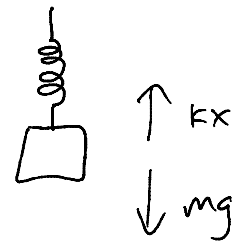
$1 = 10C_1$

$C_1 = 1/10$

$C_2 = -1/10$

$$\boxed{y = \frac{1}{10}e^{3x} - \frac{1}{10}e^{-7x} - \frac{5}{9}e^{2x}}$$

(12) $m = 0.50$ kg



$kx = mg$ at equilibrium

$k = \frac{mg}{kx} = \frac{(0.50)(9.8)}{0.392}$

$= 12.5$ N/m

$ma = -\beta v - ky + f(t)$ where β and $f(t)$ are zero

DE becomes $0.50\,\ddot{y} + 12.5\,y = 0$

aux eqn:

$0.50\,n^2 + 12.5 = 0$

$n^2 = -25$

$n = \pm 5i$ (well, $5.0i$, actually)

$= \alpha \pm \beta i$ where $\alpha = 0$, $\beta = 5$ (1)

$y = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$

$= C_1 \cos 5t + C_2 \sin 5t$ (1)

initial conditions: at $t = 0$, $y = 0$ and $y' = -1.5$ m/s

$0 = C_1 \cdot 1 + C_2 \cdot 0$ so $C_1 = 0$

$y = C_2 \sin 5t$

$y' = 5C_2 \cos 5t$

(½)

at $t = 0$ $-1.5 = 5C_2 \cdot 1$

$C_2 = -0.30$

$\boxed{y = -0.30 \sin 5t}$

(1½)

(13) A block of wood is floating in oil. A student pushes the block down 3 cm into the oil and holds it there at rest. Once the student lets go, the block then bobs up and down such that its equation of motion is $4\frac{d^2y}{dt^2}+4\frac{dy}{dt}+17y=0$, where y is the height of the block above its equilibrium position (in cm).

(5)

a) Find the height y as a function of the time elapsed after the student lets go.

b) Calculate the block's position after 1 seconds, 2 seconds, and 3 seconds have elapsed.

a) $4m^2+4m+17=0$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{16-16\cdot 17}}{8} = \frac{-4 \pm 16i}{8} = -\tfrac{1}{2} \pm 2i$$

$\alpha = -\tfrac{1}{2}$ and $\beta = 2$

$$y = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)$$

$$\boxed{= e^{-t/2}(C_1 \cos 2t + C_2 \sin 2t)}$$

$$\frac{dy}{dt} = -\tfrac{1}{2}e^{-t/2}(C_1 \cos 2t + C_2 \sin 2t) + e^{-t/2}(-2C_1 \sin 2t + 2C_2 \cos 2t)$$

at $t=0$, $y=-3$

$$-3 = \cancel{e^0}^{\,1}(C_1 \cancel{\cos 0}^{\,1} + C_2 \cancel{\sin 0}^{\,0})$$

$$-3 = C_1$$

at $t=0$, $\frac{dy}{dt}=0$

$$0 = -\tfrac{1}{2}C_1 + 2C_2$$

$$0 = \tfrac{3}{2} + 2C_2$$

$$C_2 = -\tfrac{3}{4}$$

so $y = e^{-t/2}(-3\cos 2t - \tfrac{3}{4}\sin 2t)$

$= -3e^{-t/2}(\cos 2t + \tfrac{1}{4}\sin 2t)$

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b) at $t=1s$, $y = -3e^{-1/2}(\cos 2 + \tfrac{1}{4}\sin 2) = 0.34358$ cm $= 0.3$ cm

$t=2s$, $y = -3e^{-1}(\cos 4 + \tfrac{1}{4}\sin 4) = 0.930195$ cm $= 0.9$ cm

$t=3s$, $y = -3e^{-3/2}(\cos 6 + \tfrac{1}{4}\sin 6) = -0.595969$ cm $= -0.6$ cm

make sure calculator is in rads