Math 193 – Test 2: Version A

March 12, 2018 **Name:** Solution Set
Instructor: Patricia Wrean

Total: 25 points

1. (5 points) Consider the differential equation

$$x^2y'' - xy' + y = 0$$

with solution

$$y = 3x \ln x$$

(a) State the order of this DE. 2nd ①

(b) Is this solution a general or particular solution? particular ①

(c) Show that this solution really is a solution to this DE.
Hint: Do not try to solve the DE! Just show that the above solution works.

$$y = 3x\ln x$$
$$y' = 3\ln x + 3x\cdot\frac{1}{x} = 3\ln x + 3$$
$$y'' = \frac{3}{x}$$

③

sub back into DE:

$$x^2y'' - xy' + y = 0$$
$$x^2\left(\frac{3}{x}\right) - x(3\ln x + 3) + 3x\ln x = 0$$
$$\cancel{3x} - \cancel{3x\ln x} - \cancel{3x} + \cancel{3x\ln x} = 0$$
$$0 = 0 \quad ✓$$

(−1) incorrect product rule

(−1) incorrect $\frac{d}{dx}(\ln x)$

2. (4 points) Solve, giving an explicit solution.

$$\frac{dy}{dx} - y\tan x - 3 = 0$$

linear first-order

$$\frac{dy}{dx} - y\tan x = 3$$

$$IF = e^{\int P(x)dx} = e^{\int -\tan x\, dx}$$

$$= e^{-\ln|\sec x|}$$

$$= e^{\ln(\sec x)^{-1}}$$

$$= e^{\ln \cos x}$$

$$= \cos x$$

(−1) missing negative sign

so DE is:

$$\frac{dy}{dx}\cos x - y\underbrace{\tan x\cos x}_{=\sin x} = 3\cos x$$

$$\frac{d}{dx}(y\cos x) = 3\cos x$$

$$\int d(y\cos x) = \int 3\cos x\, dx$$

$$y\cos x = 3\sin x + C$$

$$\boxed{y = 3\tan x + C\sec x}$$

$$\text{or} = \frac{3\sin x + C}{\cos x}$$

$$\text{or} = (3\sin x + C)\sec x$$

3. (5 points) Solve the following differential equation.

$$y'' - 4y' = 10\,e^{-x} - 2$$

Complementary solution:

$$m^2 - 4m = 0$$
$$m(m-4) = 0$$
$$m = 0, 4$$
$$y_c = c_1 e^{0x} + c_2 e^{4x}$$
$$\boxed{y_c = c_1 + c_2 e^{4x}}$$

particular solution: RHS $= 10e^{-x} - 2$

$$y_p = Ae^{-x} + \cancel{B} \leftarrow \boxed{\text{BAD CASE}}$$
$$y_p = Ae^{-x} + Bx$$

now differentiate:

$$y_p' = -Ae^{-x} + B$$
$$y_p'' = Ae^{-x}$$

and sub into DE:

$$y'' - 4y' = 10e^{-x} - 2$$
$$Ae^{-x} - 4(-Ae^{-x} + B) = 10e^{-x} - 2$$
$$5Ae^{-x} - 4B = 10e^{-x} - 2$$

so $5A = 10$ and $-4B = -2$

$A = 2$ $\quad$ $B = \frac{1}{2}$

$$\boxed{y_p = 2e^{-x} + \tfrac{1}{2}x}$$

$$\boxed{y = y_c + y_p = c_1 + c_2 e^{4x} + 2e^{-x} + \tfrac{1}{2}x}$$

4. (3 points) Solve the following differential equation.

$$y'' - 8y' + 17y = 0$$

aux eqn: $m^2 - 8m + 17 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{8 \pm \sqrt{64 - 4(17)}}{2}$$

$$= \frac{8 \pm \sqrt{-4}}{2} = \frac{8 \pm 2i}{2} = 4 \pm i$$

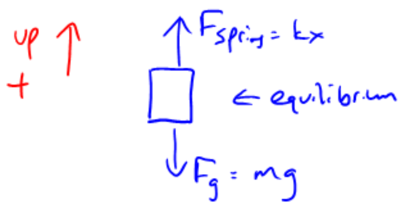
$$\boxed{y = e^{4x}(C_1 \cos x + C_2 \sin x)}$$

5. (3 points) Write down the DE and the initial conditions for the following system.

When a 2.45 N weight is suspended from a spring, the spring stretches by 0.07 m. There is an external force $F(t) = 4e^{-2t}$ but no damping. The weight is initially 0.06 m above the equilibrium position with an downwards velocity of 0.10 m/s.

downwards

DO NOT SOLVE THE DE!



$F_N = mg$ so $m = \frac{F_N}{g} = \frac{2.45N}{9.8\ m/s^2} = 0.25$ kg

$F_{spring} = F_N$ (but opposite directions)

$kx = F_N$

$k = \frac{F_N}{x} = \frac{2.45N}{0.07m} = 35$ N/m

no damping: $\beta = 0$

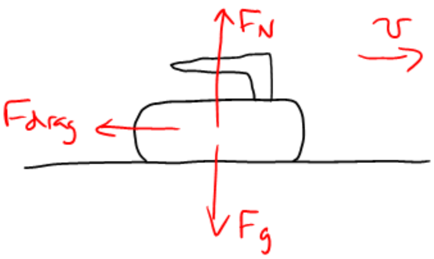
so $ma = -\cancel{\beta}^{\,0} v - kx + f(t)$

$ma + kx = f(t)$

$$\boxed{0.25 \frac{d^2x}{dt^2} + 35x = -4e^{-2t}}$$ where $$\boxed{\begin{array}{l} x(0) = 0.06m \\ \frac{dx}{dt}(0) = -0.10\ m/s \end{array}}$$

6. (5 points) After being thrown, a 20.0kg curling rock sliding along a horizontal ice surface slows down due to frictional forces which are proportional to the velocity of the rock. The initial speed of the rock is 2.5 m/s, and after 3 seconds, the rock has slowed to 1.2 m/s. How fast is the curling rock traveling after another 2 seconds have passed?

Begin with an appropriate DE, using the fact that acceleration is the rate of change of velocity.



$$\sum \vec{F} = m\vec{a}$$

$$\boxed{-kv = m\frac{dv}{dt}}$$ (separable)

$$-\frac{k}{m}\,dt = \frac{dv}{v}$$

$$-\frac{k}{m}t + C = \ln v$$

$$e^{-k/m\,t + C} = v$$

$$v = e^{-k/m\,t}e^{C}$$

$$v = C_1 e^{-k/m\,t}$$

now at $t=0$, $v = 2.5$ m/s

$$2.5 = C_1 e^{0}$$

$$C_1 = 2.5$$

at $t = 3$ seconds, $v = 1.2$ m/s

$$1.2 = 2.5\,e^{-k/20\cdot 3}$$

$$\frac{1.2}{2.5} = e^{-3k/20}$$

$$\ln\left(\frac{1.2}{2.5}\right) = \frac{-3k}{20}$$

$$k = -\frac{20}{3}\ln\left(\frac{1.2}{2.5}\right)$$

and $\frac{k}{m} = -\frac{1}{3}\ln\left(\frac{1.2}{2.5}\right)$

$$\approx 0.244656$$

so $v = C_1 e^{-k/m\,t}$

and at 5 s

$$v = 2.5\,e^{-0.244656\cdot 5}$$

$$\approx 0.735657$$

$$\boxed{v \approx 0.74 \text{ m/s}}$$