Math 193 – Test 2: Version B

March 12, 2018 **Name:** Solution Set
Instructor: Patricia Wrean

Total: 25 points

1. (5 points) Consider the differential equation

$$x^2y'' - xy' + y = 0$$

with solution

$$y = Cx \ln x$$

note: 2^{nd} order DEs need two constants for general form

(a) State the order of this DE. 2^{nd}

(b) Is this solution a general or particular solution? particular

(c) Show that this solution really is a solution to this DE.
Hint: Do not try to solve the DE! Just show that the above solution works.

$$y = Cx\ln x$$
$$y' = C\ln x + Cx \cdot \frac{1}{x} = C\ln x + C$$
$$y'' = \frac{C}{x}$$

sub back into DE:

$$x^2y'' - xy' + y = 0$$
$$x^2\left(\frac{C}{x}\right) - x(C\ln x + C) + Cx\ln x = 0$$
$$\cancel{Cx} - \cancel{Cx\ln x} - \cancel{Cx} + \cancel{Cx\ln x} = 0$$
$$0 = 0 \quad \checkmark$$

(-1) incorrect product rule

(-1) incorrect $\frac{d}{dx}(\ln x)$

2. (4 points) Solve, giving an explicit solution.

$$\frac{dy}{dx} - y\tan x - 5\sec x = 0$$

linear first-order

$$\frac{dy}{dx} - y\tan x = 5\sec x$$

$$IF = e^{\int P(x)dx} = e^{\int -\tan x\, dx}$$

$$= e^{-\ln|\sec x|}$$

$$= e^{\ln(\sec x)^{-1}}$$

$$= e^{\ln \cos x}$$

$$= \cos x$$

$\boxed{\text{(-1) missing negative in } P(x)}$

so DE is:

$$\frac{dy}{dx}\cos x - y\underbrace{\tan x\cos x}_{=\sin x} = 5\underbrace{\sec x\cos x}_{=1}$$

$$\frac{d}{dx}(y\cos x) = 5$$

$$\int d(y\cos x) = \int 5\,dx$$

$$y\cos x = 5x + C$$

$$\boxed{y = \sec x\,(5x + C)}$$

or $\frac{5x+C}{\cos x}$

3. (5 points) Solve the following differential equation.

$$y'' - 6y' = 3\,e^{-x} + 12$$

Complementary solution:

$$m^2 - 6m = 0$$
$$m(m-6) = 0$$
$$m = 0, 6$$
$$y_c = c_1 e^{0x} + c_2 e^{6x}$$
$$\boxed{y_c = C_1 + c_2 e^{6x}}$$

particular solution: RHS $= 3e^{-x} + 12$

$$y_p = Ae^{-x} + \cancel{B} \quad \leftarrow \text{BAD CASE}$$

$$y_p = Ae^{-x} + Bx$$

now differentiate:

$$y_p' = -Ae^{-x} + B$$
$$y_p'' = Ae^{-x}$$

and sub into DE:

$$y'' - 6y' = 3e^{-x} + 12$$
$$Ae^{-x} - 6(-Ae^{-x} + B) = 3e^{-x} + 12$$
$$7Ae^{-x} - 6B = 3e^{-x} + 12$$

so $7A = 3$ and $-6B = 12$

$A = 3/7$ $\quad B = -2$

$$\boxed{y_p = \frac{3}{7}e^{-x} - 2x}$$

$$\boxed{y = y_c + y_p = C_1 + c_2 e^{6x} + \frac{3}{7}e^{-x} - 2x}$$

4. (3 points) Solve the following differential equation.

$$y'' - 6y' + 10y = 0$$

aux eqn: $m^2 - 6m + 10 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{36 - 40}}{2}$$

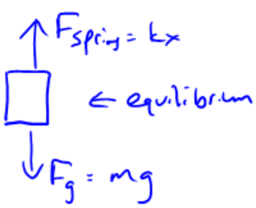
$$= \frac{6 \pm \sqrt{-4}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i$$

$$\boxed{y = e^{3x}(C_1 \cos x + C_2 \sin x)}$$

5. (3 points) Write down the DE and the initial conditions for the following system.

When a 2.45 N weight is suspended from a spring, the spring stretches by 0.07 m. There is a damping force numerically equal to three times the velocity but no external force. The weight is initially 0.10 m below the equilibrium position with an upwards velocity of 0.06 m/s.

DO NOT SOLVE THE DE!



$F_N = mg$ so $m = \frac{F_N}{g} = \frac{2.45\,N}{9.8\,m/s^2} = 0.25$ kg

$F_{spring} = F_N$ (but opposite directions)

$kx = F_N$

$k = \frac{F_N}{x} = \frac{2.45\,N}{0.07\,m} = 35\,N/m$

$f(t) = 0$

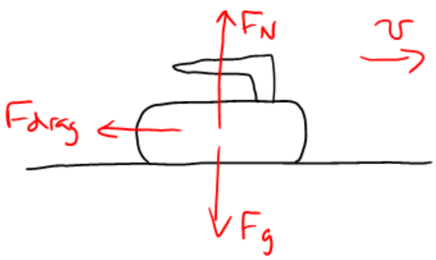
so $ma = -\beta v - kx + f(t)$

$ma + \beta v + kx = 0$

$$\boxed{0.25 \frac{d^2x}{dt^2} + 3 \frac{dx}{dt} + 35x = 0}$$ where $$\boxed{\begin{aligned} x(0) &= -0.10\text{ m} \\ \frac{dx}{dt}(0) &= 0.06\text{ m/s} \end{aligned}}$$

6. (5 points) After being thrown, an 18 kg curling rock sliding along a horizontal ice surface slows down due to frictional forces which are proportional to the square of the rock's velocity. The initial speed of the rock is 2.5 m/s, and after 3 seconds, the rock has slowed to 1.2 m/s. How fast is the curling rock traveling after another 2 seconds have passed?

   Begin with an appropriate DE, using the fact that acceleration is the rate of change of velocity.



$$\sum \vec{F} = m\vec{a}$$

$$\boxed{-kv^2 = m\frac{dv}{dt}}$$ (separable)

$$-\frac{k}{m}\,dt = \frac{dv}{v^2}$$

$$-\frac{k}{m}t + C = -\frac{1}{v}$$

$$\frac{k}{m}t - C = \frac{1}{v}$$

$$v = \frac{1}{\frac{k}{m}t - C}$$

now at $t=0$, $v = 2.5$ m/s

$$\frac{k}{m}t - C = \frac{1}{v}$$

$$-C = \frac{1}{2.5} \quad \text{and} \quad C = -0.4$$

at $t = 3$ seconds, $v = 1.2$ m/s

$$\frac{k}{18}\cdot 3 + 0.4 = \frac{1}{1.2}$$

$$\frac{k}{6} = \frac{1}{1.2} - 0.4$$

$$k = 6\left(\frac{1}{1.2} - 0.4\right)$$

$$= 2.6$$

and at $t = 5$ seconds

$$v = \frac{1}{\frac{k}{m}t - C}$$

$$= \frac{1}{\frac{2.6}{18}\cdot 5 + 0.4}$$

$$= 0.891089$$

$$\boxed{v \approx 0.89 \text{ m/s}}$$