Math 193 – Test 2: Version A

February 24, 2017 Name: Solution Set
Instructor: Patricia Wrean

Total: 25 points

1. (4 points) Solve the following differential equation with the given condition.

$$(x^2+1)^2y' + 4x = 0 \qquad \text{if } \; y(1) = 3$$

$$(x^2+1)^2 \frac{dy}{dx} = -4x$$

$$\int dy = \int \frac{-4x\,dx}{(x^2+1)^2}$$

let $u = x^2+1$
$du = 2x\,dx$

$$y = \int -\frac{2\,du}{u^2}$$

$$= -2\frac{u^{-1}}{-1} + C$$

$$y = \frac{2}{x^2+1} + C$$

at $x=1,\ y=3$

$$3 = \frac{2}{1+1} + C$$

$$C = 2$$

and $\boxed{y = \frac{2}{x^2+1} + 2}$

2. (5 points) Solve the linear differential equation. Give an explicit solution.

$$x^2dy - e^{2x}dx + 2y\,x\,dx = 0$$

$$x^2dy + 2xy\,dx = e^{2x}dx$$

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{e^{2x}}{x^2} \qquad (1/2)$$

$$IF = e^{\int P(x)dx}$$
$$= e^{\int 2/x\,dx}$$
$$= e^{2\ln x} \qquad (1)$$
$$= e^{\ln x^2}$$
$$= x^2 \qquad (1)$$

$$\frac{dy}{dx} \cdot x^2 + 2xy = e^{2x}$$

$$\frac{d}{dx}(yx^2) = e^{2x} \qquad (1)$$

$$\int d(yx^2) = \int e^{2x}dx$$

$$yx^2 = \frac{1}{2}e^{2x} + C \qquad (1)$$

$$\boxed{y = \frac{e^{2x}}{2x^2} + \frac{C}{x^2}} \qquad (1/2)$$

3. (5 points) For the function $z = y^2 e^{3x} - \sin y$, evaluate the following.

(a) $\boxed{\dfrac{\partial z}{\partial x} = 3y^2 e^{3x}}$

1

(b) $\dfrac{\partial^2 z}{\partial y^2}$

$\dfrac{\partial z}{\partial y} = 2ye^{3x} - \cos y$

$\boxed{\dfrac{\partial^2 z}{\partial y^2} = 2e^{3x} + \sin y}$

2

(c) $\dfrac{\partial^2 z}{\partial x \partial y}$

$\dfrac{\partial z}{\partial y} = 2ye^{3x} - \cos y$

$\boxed{\dfrac{\partial z^2}{\partial x \partial y} = 6ye^{3x}}$

2

4. (3 points) Evaluate.

$$\int_0^1 \int_y^{\sqrt{y}} 8x^3 dx\, dy$$

$$\int_0^1 \left[\int_y^{\sqrt{y}} 8x^3 dx \right] dy = \int_0^1 \left[\frac{8}{4} x^4 \Big|_{x=y}^{x=\sqrt{y}} \right] dy$$

$$= \int_0^1 \left[2x^4 \Big|_{x=y}^{x=\sqrt{y}} \right] dy$$

$$= \int_0^1 (2y^2 - 2y^4) dy$$

$$= \left(\frac{2y^3}{3} - \frac{2}{5} y^5 \right) \Big|_0^1$$

$$= \frac{2}{3} - \frac{2}{5}$$

$$= \frac{10}{15} - \frac{6}{15}$$

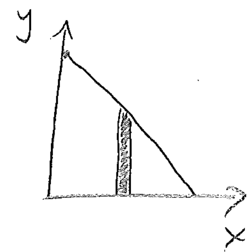
$$= \frac{4}{15} \quad (0.2\overline{6} \text{ if you insist})$$

5. (3 points) Set up but DO NOT EVALUATE the first-octant volume under the plane $x + y + z - 4 = 0$.

let $z = 0$;

then $x + y - 4 = 0$

$y = 4 - x$



$0 \le x \le 4$ ← outer

$0 \le y \le 4 - x$ ← inner

$$V = \int_V f(x,y)\, dA$$

$$= \int_0^4 \int_0^{4-x} f(x,y)\, dy\, dx$$

$$\boxed{V = \int_0^4 \int_0^{4-x} (4 - x - y)\, dy\, dx}$$

since

$x + y + z - 4 = 0$

$z = 4 - x - y$

6. (5 points) The rate of change in the intensity I of light below the surface of the ocean with respect to the depth y is proportional to I. Let I_0 be the intensity of light at the surface of the ocean. If the intensity at 5.0 m is 50% of I_0, what is the intensity at a depth of 18 m?

Start with an appropriate DE and show all of your work.

1 $$\boxed{\frac{dI}{dy} = kI}$$

(could also use $-k$ here, since light is decreasing in intensity)

$$\frac{dI}{I} = k\,dy$$

$$\ln|I| = ky + C$$

$$I = e^{ky+C} = e^{ky}e^{C} = \boxed{C_1 e^{ky}}$$ 1

but at $y=0$, $I = I_0$

$$I_0 = C_1 e^{k\cdot 0} \quad \text{so} \quad C_1 = I_0$$

and $\boxed{I = I_0 e^{ky}}$ 1

at $y = 5$, $I = 0.5\,I_0$

$$0.5\,I_0 = I_0 e^{k\cdot 5}$$

$$0.5 = e^{5k}$$

$$\ln 0.5 = 5k \approx -0.138629$$

$$\boxed{k = \tfrac{1}{5}\ln 0.5}$$ 1

what is I at $y = 18$ m?

$$I = I_0 e^{ky}$$

$$= I_0 e^{(\frac{1}{5}\ln 0.5)(18)}$$

$$\approx 0.0824692\ I_0$$

$$\boxed{= 0.08\ I_0}$$ 1