Name: ________________________

Math 251
Practice Final

Total = $\overline{}$ 70

- **Show your work. All of the work on this test must be your own.**
- You may use a scientific calculator. You may **not** use a calculator with graphing capability or a smartphone app.

GOOD LUCK!

1. (5 points) Consider the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

(a) Find the angle, $0^\circ \leq \theta \leq 180^\circ$, between the vectors $\mathbf{u}$ and $\mathbf{v}$.

(b) Compute $\|2\mathbf{u} - 3\mathbf{w}\|$.

(c) Compute $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

2. (6 points) Consider the three points

$$A = (1, 3, 1), \quad B = (2, 1, 3), \quad C = (4, 2, -1).$$

(a) Find an equation for the plane containing the three points A, B, and C.

(b) Find the area of triangle ABC.

(c) Find parametric equations for the line passing through points A and B.

3. (3 points) Consider the following vectors in $\mathbb{R}^4$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 3 \\ 4 \end{bmatrix}$$

Are $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent or linearly dependent? Justify your answer.

4. (5 points) Consider the following linear system.

$$\begin{cases} x + 2y + 3z = 5 \\ 2x - y + 2z = 3 \\ 4x + 3y + 8z = 13 \end{cases}$$

(a) Find all solutions of the system. Write your answer in column vector form.

(b) Find the values of a and b if $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ 3 \\ b \end{bmatrix}$ is a solution of the system.

5. (4 points) Solve the following matrix equation for X and simplify. You may assume that all matrices A, B, and X are invertible.

$$(AX^{-1})^{-1} = A^{-1}IA\,(AB^T)^T B^{-1}$$

6. (4 points) Consider the following two matrices.

$$A = \begin{bmatrix} x & 2 \\ 3 & 1+2x \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & x & 2 \\ 1 & 3 & x-5 \end{bmatrix}$$

Find all values of x for which

$$\det(A) = \det(B).$$

7. (5 points) Consider the matrix $A = \begin{bmatrix} 4 & 1 & -1 \\ 12 & 5 & -2 \\ -4 & 3 & 6 \end{bmatrix}$ and column vector $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ -16 \end{bmatrix}$.

   (a) Find an LU factorization of A.

   (b) Use your answer to (a) to solve $A\mathbf{x} = \mathbf{b}$.

8. (6 points)

   (a) Find the matrix corresponding to the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects a vector through the y-axis.

   (b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates a vector by 60° counterclockwise followed by a reflection through the the y-axis. Evaluate

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right).$$

9. (5 points) Find all complex numbers solutions of the following equations.

   (a) $z^2 - 8z + 25 = 0$

   (b) $z^3 + 27 = 0$

   Express your answers in the form $a + bi$.

10. (6 points) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}.$$

(a) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(b) Evaluate A^8.

11. (5 points) Ticket prices for an IMAX movie at the local movie theatre are \$20 for adults, \$15 for seniors, and \$10 for children. On one night in February, there were 150 total tickets sold for \$2505 .

   (a) Set up and solve a system of equations to find how many of each type of ticket were sold on that night. Leave your answer in parametric form.

   (b) Find the maximum and minimum values for the number of each type of ticket sold that night.

12. (6 points) Given that

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 & 4 \\ 2 & 1 & 1 & 1 & 5 \\ 1 & 4 & -5 & 2 & 7 \\ 3 & 0 & 4 & 1 & 6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 4 & 0 \\ -2 & 1 & -5 & 4 \\ 1 & 1 & 2 & 1 \\ 4 & 5 & 7 & 6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for each of the four subspaces: Row(A), Col(A), Null(A), and Null(A^T).

(b) Verify from your basis vectors found in part (a) that

$$\text{Row}(A) \perp \text{Null}(A) \quad \text{and} \quad \text{Col}(A) \perp \text{Null}(A^T).$$

13. (5 points) Consider the following linear system.

$$\begin{cases} x_1 + 3x_2 = 2 \\ 4x_1 + x_2 = 1 \\ 2x_1 + x_2 = 3 \end{cases}$$

(a) Show that the system has no solutions.

(b) Find the least-squares solution of the system.

14. (5 points) Let

$$W = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\1\end{bmatrix}\right) \quad \text{and} \quad \mathbf{v} = \begin{bmatrix}3\\2\\5\end{bmatrix}.$$

Find $\operatorname{proj}_W(\mathbf{v})$ and $\operatorname{perp}_W(\mathbf{v})$.