Section 2.2: Direct Methods for solving Linear Systems

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definition: A matrix is in row-echelon form if

1. any rows with only zeros are at the bottom
2. in each non-zero row, the first non-zero entry is in a column to the left of any non-zero entries below it

examples:

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & -2 & 7 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Gaussian elimination: to solve a system of linear equations, transform its augmented matrix to row-echelon form and use back substitution to solve

example: use Gaussian elimination to solve:

$$\begin{cases} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 6y - 5z = 0 \end{cases}$$

answer:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ R_2 - 2R_1 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

$$\begin{array}{r} \\ 3R_2 \\ 2R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 6 & -21 & -51 \\ 0 & 6 & -22 & -54 \end{array}\right]$$

$$\begin{array}{r} \\ \frac{1}{3} R_2 \\ R_3 - R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

this corresponds to

$$\begin{cases} x + y + 2z = 9 \\ \quad 2y - 7z = -17 \\ \qquad\quad z = 3 \end{cases}$$

use back substitution:

$$\begin{aligned} 2y - 7z &= -17 \\ 2y - 21 &= -17 \\ 2y &= 4 \\ y &= 2 \end{aligned}$$

$$\begin{aligned} x + y + 2z &= 9 \\ x + 2 + 6 &= 9 \end{aligned}$$

$x = 1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

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definition:

leading variables: variables corresponding to leading non-zero entries in the associated matrix

free variables: all other variables

examples: for the system with the matrix

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

x is leading
y is leading
z is free

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

all x, y, z leading

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 8 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x leading
y free
z leading