Section 3.3: The Inverse of a Matrix

Wednesday, October 10, 2018 11:46 AM

definition: The inverse of an $N \times N$ matrix A is an $N \times N$ matrix A^{-1} such that

$$AA^{-1} = I_N$$

$$A^{-1}A = I_N$$

note: matrix must be square to have an inverse

but not all square matrices have an inverse

(however, if it does have an inverse, that inverse is unique)

Special case: 2×2 matrices

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

the determinant of A is

$$\det(A) = ad - bc$$

then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

note: A^{-1} only exists if $\det(A) \neq 0$

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example: find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

answer: $\det(A) = ad - bc = 1(4) - 2(3) = -2$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

note: optional check:

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

why do we care?

Suppose $AB = C$ where A, B, and C are matrices and A and C are known

- we want to find B

long, horrible method #1:

DON'T USE THIS!

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{find this matrix}} = \begin{bmatrix} 4 & 5 \\ 12 & 9 \end{bmatrix}$$

then
$$\begin{aligned} a + 2c &= 4 \\ b + 2d &= 5 \\ 3a + 4c &= 12 \\ 3b + 4d &= 9 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 3 & 0 & 4 & 0 & 12 \\ 0 & 3 & 0 & 4 & 9 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

So the unknown matrix is

$$B = \begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix}$$

or somewhat less annoying method #2:

$AB = C$ where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 5 \\ 12 & 9 \end{bmatrix}$

find B

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answer:

$$\underbrace{A^{-1}A}B = A^{-1}C$$

$$I_2 B = A^{-1}C$$

$$B = A^{-1}C$$

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

so $B = A^{-1}C = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 12 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix}$$

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example: solve the system using a matrix inverse

$$\begin{cases} 3x + 2y = 1 \\ 6x - y = 17 \end{cases}$$

answer: $\underbrace{\begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 17 \end{bmatrix}}_{\vec{b}}$

now solve for $\vec{x}$:

$$A\vec{x} = \vec{b}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$I_2\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

so find A^{-1}: $\det(A) = 3(-1) - 2(6) = -15$

$$A^{-1} = -\frac{1}{15}\begin{bmatrix} -1 & -2 \\ -6 & 3 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b} = \frac{1}{15}\begin{bmatrix} 1 & 2 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 17 \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} 35 \\ -45 \end{bmatrix}$$

$$= \begin{bmatrix} 7/3 \\ -3 \end{bmatrix} \quad \text{or} \quad \frac{1}{3}\begin{bmatrix} 7 \\ -9 \end{bmatrix}$$