Section 3.4: The LU Factorization

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$$\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 2 & -1 & 6 \end{bmatrix}$$ lower triangular matrix

$$\begin{bmatrix} 2 & 6 & 1 \\ 0 & 3 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$ upper triangular matrix

for a linear system

$$A\vec{x} = \vec{b}$$

where A is a square matrix, we want to express A as

$$A = LU$$ where L: lower triangular matrix
U: upper triangular matrix

note: L and U are not unique

note: textbook goes further and it insists that L be unit lower triangular

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$ ← main diagonal entries are ones

then to solve for $\vec{x}$:

$$A\vec{x} = \vec{b}$$

$$LU\vec{x} = \vec{b}$$

$$L(U\vec{x}) = \vec{b}$$

now call $U\vec{x} = \vec{y}$

$$L\vec{y} = \vec{b}$$

recall:

$$\begin{bmatrix} 5 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 7 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \\ 21 \end{bmatrix}$$

having a lower triangular matrix means you can use back substitution to find $\vec{y}$ easily

then do $U\vec{x} = \vec{y}$

so
$$\begin{bmatrix} 8 & 1 & 7 \\ 0 & 5 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

and use back substitution to solve for $\vec{x}$

example: solve the system $A\vec{x} = \vec{b}$ where example: solve the system $A\vec{x} = \vec{b}$ where

$$A = \overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -2 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ -3 \\ -1 \\ 0 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$L\underbrace{U\vec{x}}_{\vec{y}} = \vec{b}$$

$$L\vec{y} = \vec{b}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -1 \\ 0 \end{bmatrix}$$

so $y_1 = 1$

$-2y_1 + y_2 = -3$, so $-2(1) + y_2 = -3$, and $y_2 = -1$

$3y_1 - 2y_2 + y_3 = -1$, so $3 + 2 + y_3 = -1$, and $y_3 = -6$

$-5y_1 + 4y_2 - 2y_3 + y_4 = 0$, so $-5 - 4 + 12 + y_4 = 0$, and $y_4 = -3$

$$U\vec{x} = \vec{y}$$

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -6 \\ -3 \end{bmatrix}$$

$x_1 + 4x_2 + 3x_3 = 1$, $x_1 - \frac{40}{3} + 9 = 1$, $x_1 = 16/3$

$$x_1 + 4x_2 + 3x_3 = 1, \qquad x_1 - \frac{40}{3} + 9 = 1 \qquad x_1 = 16/3$$

$$3x_2 + 5x_3 + 2x_4 = -1, \qquad \text{so} \quad 3x_2 + 15 - 6 = -1 \qquad \text{and} \quad x_2 = -10/3$$

$$-2x_3 = -6, \qquad x_3 = 3$$

$$x_4 = -3$$

$$x_1 = 1 - 9 + \frac{40}{3}$$

$$= \frac{3}{3} - \frac{27}{3} + \frac{40}{3} = \frac{16}{3}$$

$$\text{so} \quad \vec{x} = \begin{bmatrix} 16/3 \\ -10/3 \\ 3 \\ -3 \end{bmatrix}$$