Section 3.5: cont'd

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theorem: if A is $M \times N$,

N = number of variables
= # leading + # free
= Rank(A) + Nullity(A)

example:

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

note: already in RREF

Rank(A) = 3 (= dim of Row(A) = dim of Col(A))

Nullity(A) = 1 (= number of free variables)

note: if A is a square matrix, then Rank(A) = Rank(A^T)

proof: Row(A) = Col(A^T)

Rank(A) = dim of Row(A)
= dim of Col(A^T)
= Rank(A^T)

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example: Find a basis of $S = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ in $\mathbb{R}^5$ where

$$\vec{v}_1 = [1 \quad -2 \quad -3 \quad 2 \quad -4]$$

$$\vec{v}_2 = [\,-3 \quad 7 \quad -1 \quad 1 \quad -3\,]$$
$$\vec{v}_3 = [\,2 \quad -5 \quad 4 \quad -3 \quad 7\,]$$
$$\vec{v}_4 = [\,-3 \quad 6 \quad 9 \quad -6 \quad 1\,]$$

method #1:

$$A = \begin{bmatrix} 1 & -2 & -3 & 2 & -4 \\ -3 & 7 & -1 & 1 & -3 \\ 2 & -5 & 4 & -3 & 7 \\ -3 & 6 & 9 & -6 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -23 & 16 & 0 \\ 0 & 1 & -10 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$S = \text{row}(A)$

and the basis of row(A) is the set of the non-zero rows of A

$$\{\, [1 \quad 0 \quad -23 \quad 16 \quad 0],\ [0 \quad 1 \quad -10 \quad 7 \quad 0],\ [0 \quad 0 \quad 0 \quad 0 \quad 1] \,\}$$

method #2: if for some reason, we want our basis to consist of some of the original vectors, then we need to do the following

$$A^T = \begin{bmatrix} 1 & -3 & 2 & -3 \\ -2 & 7 & -5 & 6 \\ -3 & -1 & 4 & 9 \\ 2 & 1 & -3 & -6 \\ -4 & -3 & 7 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

note: $\vec{v}_3 = -\vec{v}_1 - \vec{v}_2$

$S = \text{col}(A^T)$

the basis is $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$

theorem: let S be a subspace of $\mathbb{R}^n$ and let $\beta = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ be a basis for S.

then any vector $\vec{u}$ in S can be uniquely expressed as a linear combination

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_k\vec{v}_k$$

notation $[\vec{u}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$ are the coordinates of $\vec{u}$ with respect to β

note: $\vec{v} = 3\hat{i} + 2\hat{j} - 5\hat{k}$

$$= \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

as $\beta = \{\hat{i}, \hat{j}, \hat{k}\}$ is a basis of $\mathbb{R}^3$

example: let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

a) show that $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$

show that the vectors are LI

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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so $\mathcal{B}$ is a basis

b) find the coordinates of $\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ with respect to $\mathcal{B}$.

answer:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 0 & 1 & 3 \\ 0 & 3 & 1 & 4 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 13 \end{array}\right]$$

so $\vec{u} = -5\vec{v}_1 - 3\vec{v}_2 + 13\vec{v}_3$

$$\vec{u} = \begin{bmatrix} -5 \\ -3 \\ 13 \end{bmatrix}_{\mathcal{B}}$$