Section 4.1: Introduction to Eigenvalues and Eigenvectors

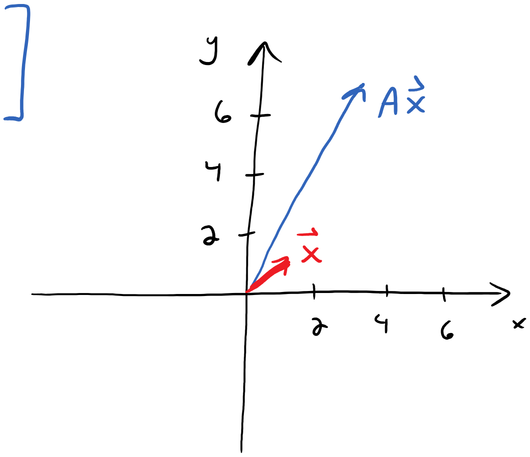
Tuesday, November 6, 2018 4:24 PM

Let A be a 2×2 matrix.

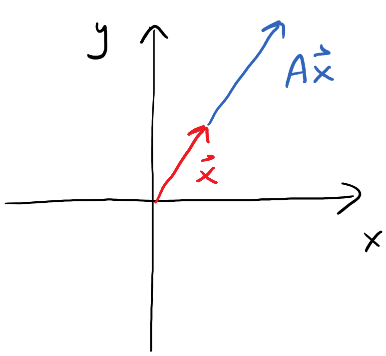
Typically $A\vec{x}$ scales and rotates $\vec{x}$.

example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

then $A\vec{x} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$



We are interested in the special vectors that are simply scaled by A and <u>not</u> rotated:



definition: let A be an $N\times N$ matrix

a scalar λ is called an eigenvalue of A

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Greek character "lambda"

if there is a non-zero vector $\vec{x}$ such that

$$A\vec{x} = \lambda \vec{x}$$

In this case, $\vec{x}$ is an eigenvector corresponding to λ.

note: If $A\vec{x} = \lambda\vec{x}$ then $A\vec{x}$ and $\vec{x}$ have the same (or opposite) directions and A simply scales $\vec{x}$.

example: let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. Show that $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors of A.

answer: $A\vec{x}_1 = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\vec{x}_1$

compare

$\lambda_1 = 2$

$A\vec{x}_2 = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\vec{x}_2$

$\lambda_2 = 3$

But how can you find λ if you don't know $\vec{x}$?

$$A\vec{x} = \lambda\vec{x}$$

$$A\vec{x} - \lambda\vec{x} = 0$$

$$A\vec{x} - \lambda I_2\vec{x} = 0$$

$$(A - \lambda I_2)\vec{x} = 0$$

$$\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we don't want this to reduce to I_2 because in that case the only solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

and we are interested in non-zero vectors

so we want the matrix $(A - \lambda I_2)$ to <u>not</u> have an inverse - we want the determinant to equal zero

so $$\boxed{\det(A - \lambda I_2) = 0}$$

- this gives us a polynomial whose roots are the eigenvalues of A

example: Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

answer: $A - \lambda I_2 = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix}$$

$$\det(A - \lambda I_2) = 0$$

$$(1-\lambda)(4-\lambda) - 2(-1) = 0$$

$$4 - 5\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 3)(\lambda - 2) = 0$$

$$\lambda = 2, 3$$