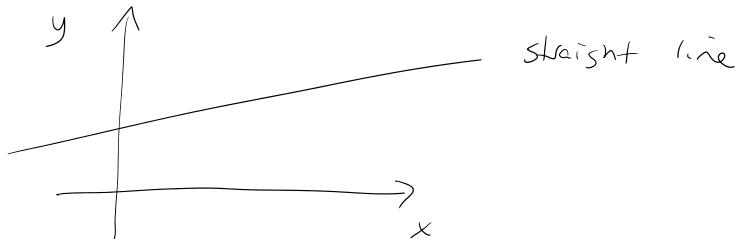


## Section 1.3: Lines and Planes

Wednesday, September 13, 2023 10:00 AM

### Lines in $\mathbb{R}^2$

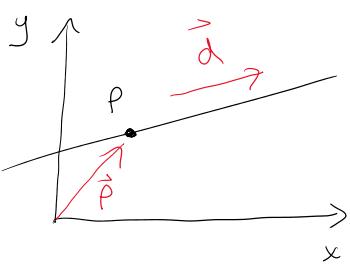
consider a line in  $\mathbb{R}^2$ :



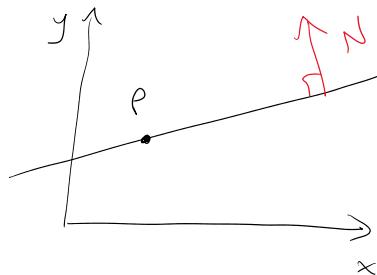
what information do we need to specify this particular line

- slope and y-intercept (except for vertical lines)
- two points on line (must be two different points)

or if we think in terms of vectors



some point on line  
and direction  
vector  $\vec{d}$



some point  $P$  on  
the line and  
normal vector  $\vec{N}$   
where  $\vec{N}$  is perpendicular  
to line

then all points on the  
line are found by  
adding a multiple of  
 $\vec{d}$  to vector  $\vec{p}$

and any other point  $\vec{x}$   
on the line has

$$\vec{px} = t \vec{d}$$

then any point  $X$  on  
the line has

$$\vec{px} \perp \vec{N}$$

$$\vec{px} \cdot \vec{N} = 0$$

some multiple  
of  $\vec{d}$

Let's look at  $\vec{P}X \cdot \vec{N} = 0$  where  $X = (x, y)$   
 (normal form of line)  $\begin{bmatrix} x - p_1 \\ y - p_2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0$   $\vec{N} = \begin{bmatrix} a \\ b \end{bmatrix}$

so  $\begin{bmatrix} x - 3 \\ y + 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 0$

↑  
 this is a line through point  $P = (3, -1)$  and normal  
 $\vec{N} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

now take dot product

$$(x - p_1)a + (y - p_2)b = 0$$

general equation of a line

so  $(x - 3)2 + (y + 1)5 = 0$

$$2x + 5y = 1$$

note:  $2x + 5y = 1$  general form

$$\begin{aligned} 5y &= -2x + 1 \\ y &= \left(-\frac{2}{5}\right)x + \frac{1}{5} \end{aligned}$$

slope is  $-\frac{2}{5}$

perpendicular is negative reciprocal

perpendicular to line has slope

$$\frac{5}{2}$$

$\alpha$

Section 1.3: cont'd    2023/09/15

example: find the general equation of the line perpendicular to  $\vec{N} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ , passing through the point  $P = (1, 4)$

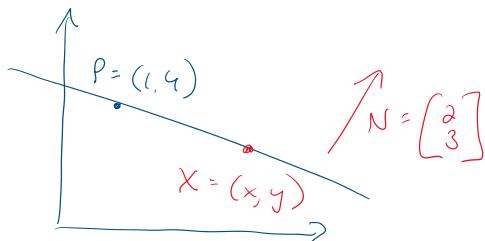
answer:

$$\vec{P}\vec{X} \cdot \vec{N} = 0$$

$$\begin{bmatrix} x-1 \\ y-4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0$$

$$2(x-1) + 3(y-4) = 0$$

$$2x + 3y = 14 \quad \text{general form}$$



note: the general form looks like

$$Ax + By = C \quad \text{where } A, B, C \text{ real}$$

and if possible  $A, B, C$  are integers  
with  $A$  positive

you sometimes see

$$Ax + By + D = 0$$

note: shortcut for the lazy:

$$P = (1, 4)$$

$$\vec{N} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\text{general form } Ax + By = C$$

$$2x + 3y = C$$

$$\text{plug in } P: 2(1) + 3(4) = C \\ C = 14$$

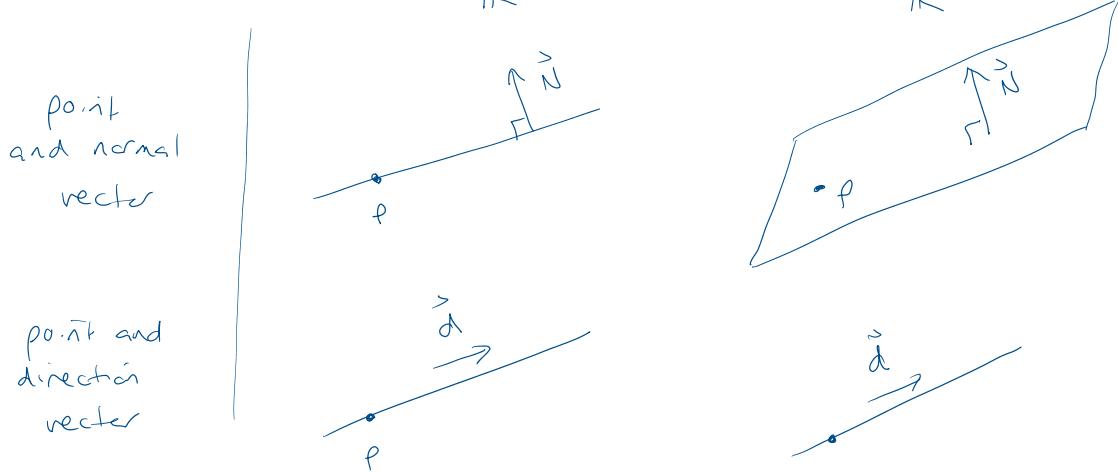
$$2x + 3y = 14$$

in  $\mathbb{R}^2$ , we'd say that you can define a line by

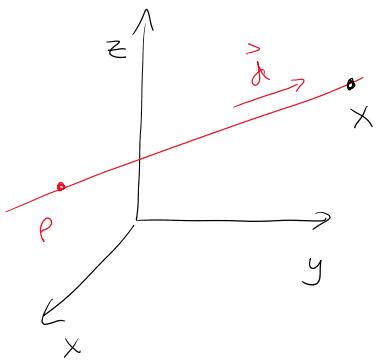
in  $\mathbb{R}^2$ , we'd said that you can define a line by a point  $P$  and direction vector  $\vec{d}$ , where  $\vec{d}$  is parallel to the line

The nice thing is that this also works in  $\mathbb{R}^3$ :

$\mathbb{R}^2$  vs  $\mathbb{R}^3$



lines in  $\mathbb{R}^3$ :



given point  $P = (P_1, P_2, P_3)$   
and direction vector

$$\vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

where  $\vec{d}$  is parallel to the line

then let  $X$  be arbitrary point on the line

$$X = (x, y, z)$$

so

$$\vec{PX} \parallel \vec{d}$$

$$\vec{PX} = t \vec{d} \quad \text{where } t \text{ is a scalar}$$

then

$$\begin{bmatrix} x - p_1 \\ y - p_2 \\ z - p_3 \end{bmatrix} = t \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

vector equation  
of a line

or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

equivalent

finally

$$\left\{ \begin{array}{l} x = p_1 + t d_1 \\ y = p_2 + t d_2 \\ z = p_3 + t d_3 \end{array} \right.$$

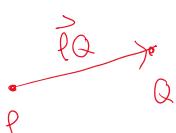
parametric  
equations  
of a line

example: find parametric equations for the line joining points  $P = (1, 4, -2)$  and  $Q = (2, 3, 5)$ .

$$\vec{PQ} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$$

then  $\vec{PQ} = t \vec{PQ}$

$$\begin{bmatrix} x - 1 \\ y - 4 \\ z + 2 \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$$



$$\left\{ \begin{array}{l} x = 1 + t \\ y = 4 - t \\ z = -2 + 7t \end{array} \right.$$

note: you could use  $Q$  as your point  
and/or  $\vec{QP}$  as your vector  
(or use any non-zero multiple of  $\vec{PQ}$ )

- there is another way to describe a line in 3D  
- it's the intersection of two planes  
(but we haven't done planes yet, so  
we'll have to come back to this one)

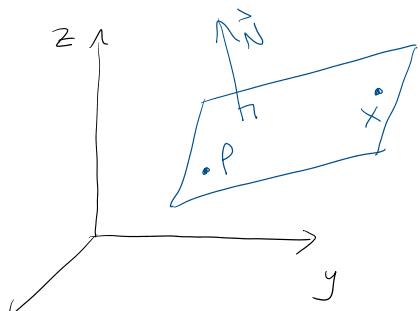
planes in  $\mathbb{R}^3$ :

we can describe a plane in  $\mathbb{R}^3$  using a point  $P = (P_1, P_2, P_3)$  and a vector

$$\vec{N} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \quad \text{perpendicular to the plane}$$

[ note: in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , can use the word perpendicular. In  $\mathbb{R}^3$  and up, use the word orthogonal. ]

we want an equation containing arbitrary point  
 $X = (x, y, z)$



$$\text{so } \vec{PX} \perp \vec{N}$$

$$\vec{PX} \cdot \vec{N} = 0$$

$$\boxed{\begin{bmatrix} x - P_1 \\ y - P_2 \\ z - P_3 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0}$$

normal form of plane

$$n_1(x - P_1) + n_2(y - P_2) + n_3(z - P_3) = 0$$

$$n_1x + n_2y + n_3z = \underbrace{n_1P_1 + n_2P_2 + n_3P_3}_{\text{a constant}}$$

general form

$$\boxed{n_1x + n_2y + n_3z = D}$$

the general equation of a plane

$$Ax + By + Cz = D$$

$A$   $B$   $C$

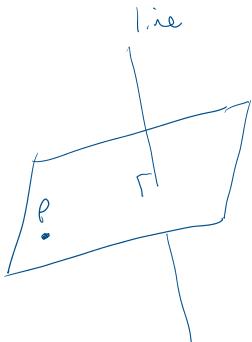
These coefficients are components of normal

so  $2x + 3y - 5z = 8$  has normal  $\vec{N} = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$

example: Find the equation of a plane through the point  $P = (1, 3, 2)$  and perpendicular to the line given by

$$\begin{cases} x = 2 + 3t \\ y = -1 + t \\ z = 2 - 4t \end{cases}$$

answer:



so the line is parallel to the normal

and we can use any vector from this line as the normal

how do we find that vector?

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$$

$$\text{so } \vec{N} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$$

but if you don't see that, then find two points on the line, then find the vector between them

$$\text{so let } t=0, \quad A = (2, -1, 2) \\ t=1, \quad B = (5, 0, -2)$$

$$\vec{AB} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$$

$$\text{then } \vec{P} \vec{x} \cdot \vec{N} = 0$$

$$\begin{bmatrix} x - 1 \\ y - 3 \\ z - 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = 0$$

← one perfectly acceptable answer (normal form)

or to get general form:

$$Ax + By + Cz = D$$

$$3x + y - 4z = D$$

plug in point  $(1, 3, 2)$ :

$$3(1) + 1(3) - 4(2) = D$$

$$D = -2$$

$$3x + y - 4z = -2$$

general form

Section 1.3: cont'd      2023/09/18

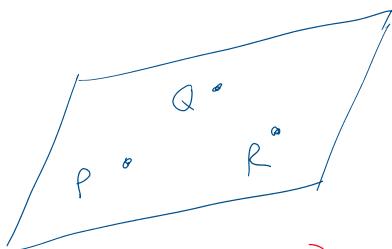
example: Find the general equation of a plane containing the points

$$P = (2, 1, 3)$$

$$Q = (1, 0, 4)$$

$$R = (3, 1, -6)$$

answer:



need normal and a point

$$\vec{PQ} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{PR} = \begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 1 \\ 1 & 0 & -9 \end{vmatrix}$$

$$= 9\hat{i} - 8\hat{j} + \hat{k} = \begin{bmatrix} 9 \\ -8 \\ 1 \end{bmatrix}$$

so eqn of plane

$$9x - 8y + z = d$$

plus in any point (I'll use P)

$$9(2) - 8(1) + 3 = d$$

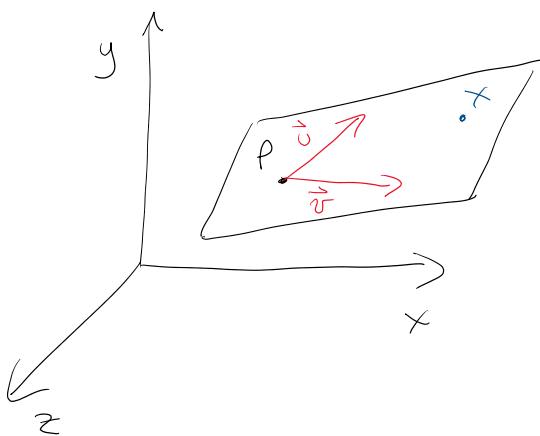
$$d = 13$$

$$\boxed{9x - 8y + z = 13}$$

optional check with point Q:  $9(1) - 8(0) + (4) = 13 \quad \checkmark$   
R:  $9(3) - 8(1) - (6) = 13 \quad \checkmark$

---

parametric equations for a plane



given point  $P$  in the plane and two vectors  $\vec{u}$  and  $\vec{v}$  parallel to the plane, where  $\vec{u}$  and  $\vec{v}$  are not parallel to each other,

this defines a plane

let  $X = (x, y, z)$  be some arbitrary point in the plane

then  $\vec{PX} = t\vec{u} + s\vec{v}$

$$\boxed{\vec{x} = \vec{P} + t\vec{u} + s\vec{v}}$$

vector form for a plane

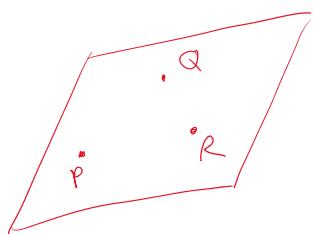
the vector from  
the origin to  
point  $x$  in the plane

if you then write out the individual equations  
you get the parametric equations

example: Find parametric equations for the plane containing

$$\begin{aligned} P &= (2, 1, 3) \\ Q &= (1, 0, 4) \\ R &= (3, 1, -6) \end{aligned}$$

answer:



need a point in the plane

and two vectors in the plane

$$\vec{P} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \vec{PQ} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{PR} = \begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}$$

$$\text{then } \vec{x} = \vec{P} + t\vec{PQ} + s\vec{PR}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix} \quad \text{vector form}$$

finally:

$$\boxed{\begin{cases} x = 2 - t + s \\ y = 1 - t \\ z = 3 + t - 9s \end{cases}}$$

parametric  
form

note: these equations are not unique

any point and any two vectors parallel  
to the plane (but not each other)  
will work

also, earlier we found that the general equation  
of this line

also, earlier we found that the general equation of this line

$$9x - 8y + z = 13$$

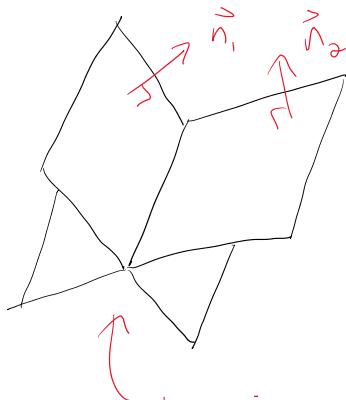
$$3(2-t+s) - 8(1-t) + (3+t-9s) = 13$$

$$\cancel{18} - \cancel{9t} + \cancel{9s} - \cancel{8} + \cancel{8t} + \cancel{3} + \cancel{t} - \cancel{9s} = 13$$

$$13 = 13$$



Line in  $\mathbb{R}^3$  as Intersection of two planes



The intersection of these two planes is the line that we want

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1, \\ a_2 x + b_2 y + c_2 z = d_2 \end{cases}$$

so I can find the direction vector by cross product of  $\vec{n}_1$  and  $\vec{n}_2$

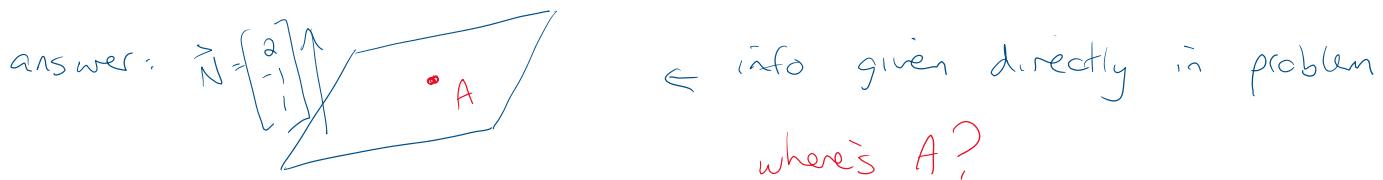
how do you find a point on the line? see section 2.2

distances: use projections to find the distance from

- a point to a line
- a point to a plane

example: find the distance between point  $B = (3, -1, 2)$  and the plane  $2x - y + z = 4$ .

Also, find the closest point to  $B$  in the plane!  
 $\bullet B = (3, -1, 2)$



first: notice that  $\vec{AB}$  will be parallel to  $\vec{N}$

second: pick some other point on the plane  $P$

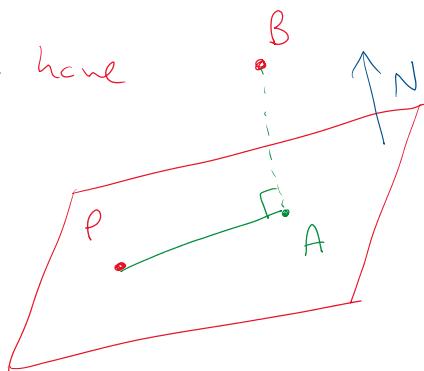
what's the easiest way? set two coords to zero and find the third:

$$2x - y + z = 4$$

$$\text{set } x = y = 0, \text{ then } z = 4$$

$$P = (0, 0, 4)$$

so now we have



now the distance we want is  $\|\vec{AB}\|$

but what's  $\vec{AB}$ ?

$$\vec{AB} = \text{proj}_{\vec{N}} (\vec{PB})$$

where  $\vec{PB} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$  and  $\vec{N} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$$\vec{AB} = \text{proj}_{\vec{N}} (\vec{PB}) = \frac{\vec{N} \cdot \vec{PB}}{\vec{N} \cdot \vec{N}} \vec{N}$$

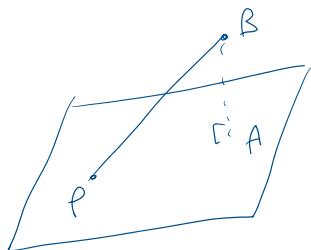
$$tB = \text{proj}_{\vec{N}}(PB) = \frac{\vec{N} \cdot tB}{\vec{N} \cdot \vec{N}} \vec{N}$$

$$= \frac{6+1-2}{4+1+1} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$= \frac{5}{6} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{distance} = \|\vec{AB}\| = \frac{5}{6} \sqrt{4+1+1} = \frac{5}{6} \sqrt{6}$$

okay, so where is point A?



given  $P$ ,  $B$ ,  $\vec{PB}$  and  $\vec{AB}$

$$\vec{AB} = \vec{B} - \vec{A}$$

$$\text{so } \vec{A} = \vec{B} - \vec{AB}$$

$$= \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \frac{5}{6} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

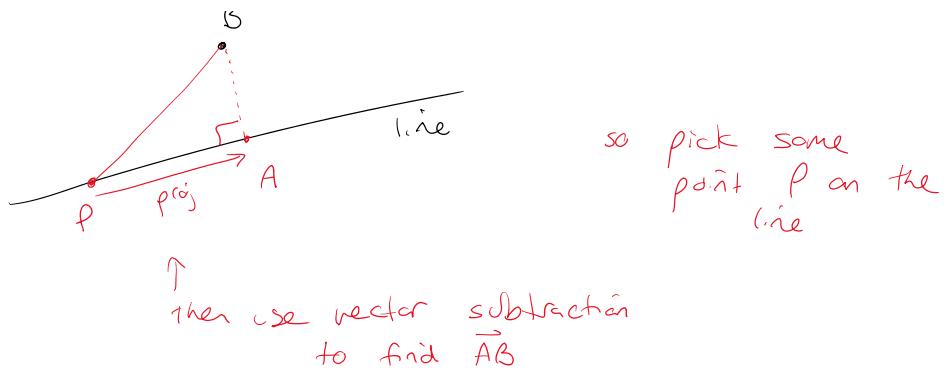
$$= \begin{bmatrix} \frac{4}{3} \\ -\frac{1}{6} \\ \frac{7}{6} \end{bmatrix}$$

$$\text{so } \underline{\underline{\underline{\text{point } A}}} = \left( \frac{4}{3}, -\frac{1}{6}, \frac{7}{6} \right)$$

Section 1.3: cont'd      2023/09/18

distance from a point to a line in  $\mathbb{R}^3$

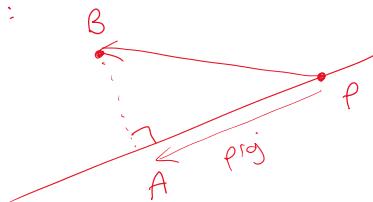




example: find the distance between the point  $B = (1, -1, 2)$  and the line

$$\left\{ \begin{array}{l} x = 3 + t \\ y = -2 - 2t \\ z = 4 + 2t \end{array} \right.$$

answer:



find some point  $P$  on the line by letting  $t=0$

$$P = (3, -2, 4)$$

the direction vector  $\vec{v}$  along the line is

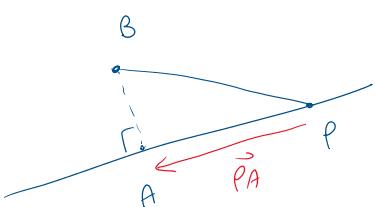
$$\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

then  $\vec{PA} = \text{proj}_{\vec{v}} (\vec{PB})$

$$\text{where } \vec{PB} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{PA} = \frac{\vec{v} \cdot \vec{PB}}{\vec{v} \cdot \vec{v}} \quad \vec{v} = \frac{-2 - 2 - 4}{1 + 4 + 4} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$= -\frac{8}{9} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$



$$\text{then } \vec{PA} + \vec{AB} = \vec{PB}$$

$$\vec{v} \quad \vec{v} \quad \vec{v}$$

$$\text{then } \vec{PA} + \vec{AB} = \vec{PB}$$

$$\vec{AB} = \vec{PB} - \vec{PA}$$

$$= \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} - (-\frac{1}{9}) \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{10}{9} \\ -\frac{7}{9} \\ -\frac{2}{9} \end{bmatrix} = -\frac{1}{9} \begin{bmatrix} 10 \\ 7 \\ 2 \end{bmatrix}$$

$$\text{distance} = \|\vec{AB}\| = \sqrt{\frac{1}{9}(10^2 + 7^2 + 2^2)}$$

$$= \frac{1}{9} \sqrt{153} = \frac{\sqrt{17}}{3} \approx 1.374$$