

## Section 3.1: Matrix Operations

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definition: a matrix is a rectangular array of numbers, called the "entries" of the matrix

the size of a matrix is  $M \times N$  ("M by N") if it has M rows and N columns

for  $3 \times 3$  matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Main diagonal

so  $a_{ij}$  is in row i  
and column j

the main diagonal entries of A are

$$a_{11}, a_{22}, a_{33}, \dots, a_{NN}$$

a square matrix has #rows = #columns

### ① matrix equality

two matrices A and B are equal if and only if they have the same size and

$$a_{ij} = b_{ij} \quad \text{for all } i \text{ and } j$$

example:  $A = \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} 6 & x \\ 9 & 5 \end{bmatrix}$

example:  $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$  and  $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

If  $A = B$ , what can we conclude?

$$a = x$$

$$x = z$$

$$y = b$$

$$z = d$$

example: Let  $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

does  $A = B$ ? No, not same size

## ② Matrix Addition/Subtraction

example: Let  $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 3 & 1 & 6 \\ 1 & 2 & 0 \end{bmatrix}$

Find  $A + B$ .

$$A + B = \begin{bmatrix} 1+3=4 & 3 & 10 \\ 3 & 5 & 1 \end{bmatrix}$$

definition  $(A + B)_{ij} = a_{ij} + b_{ij}$

$$(A - B)_{ij} = a_{ij} - b_{ij}$$

note: if  $A$  and  $B$  are not the same size,

then  $A + B$  is undefined  
(DNE = does not exist)

exist)

### (3) scalar multiplication

If  $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \end{bmatrix}$ , find  $7A$ .

$$7A = \begin{bmatrix} 7 & 14 & 28 \\ 14 & 21 & 7 \end{bmatrix}$$

definition:  $(cA)_{ij} = ca_{ij}$

### (4) Matrix multiplication

example: let  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$  and  $B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$

$AB$  will be  $3 \times 2$

3x2      same      2x2

Find  $AB$ .

answer:  $AB = \begin{bmatrix} 25 & 28 \\ 57 & 64 \\ 89 & 100 \end{bmatrix}$

$= 1(7) + 2(9) \quad 1(8) + 2(10)$

definition if  $A$  is an  $M \times N$  matrix and  $B$  is an  $N \times P$  matrix, then the product  $AB$  is defined and has size  $M \times P$

$$(AB)_{ij} = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B)$$

$\uparrow$   
dot product

example: now find  $BA$

example: now find  $BA$

$$\begin{matrix} 2 \times 2 & 3 \times 2 \\ B & A \end{matrix}$$

ONE (undefined)

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example:  $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix}$

a) find  $AB$

$$AB = \begin{bmatrix} 5 & 13 \\ 9 & 31 \end{bmatrix}$$

b) find  $BA$

$$BA = \begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 22 & 15 \\ 18 & 19 \end{bmatrix}$$

In general  $AB \neq BA$  if both products are defined

order matters

## ⑤ Identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- square matrix with ones on main diagonal, zeros everywhere else

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

example: let  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

a) Find  $I_2 A$

$$I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = A$$

b) Find  $A I_3$

$$A I_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = A$$

(e) Powers of a Square Matrix

If  $A$  is  $N \times N$

$$\text{then } A^2 = A A$$

$$A^3 = A A A = A A^2$$

$$\text{note: } A^0 = I_N$$

warning! don't take shortcuts

$$\text{if } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \text{ then } A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$\not\equiv$

## ⑦ Transpose of a Matrix

example:  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

then  $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

definition  $(A^T)_{ij} = A_{ji}$

to get  $A^T$  we change the rows of  $A$  to columns

- if  $A$  is  $M \times N$ , then  $A^T = N \times M$

definition: A square matrix is symmetric if  $A^T = A$

example:  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$  has  $A^T = A$

examples: Let  $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 5 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 4 \\ 4 & 0 \\ 3 & 1 \end{bmatrix}$ , and  $C = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

a) find  $A^T C$

$$A^T C = \begin{bmatrix} 2 & 4 \\ 3 & 0 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 14 \\ 6 & 3 \\ 2 & 16 \end{bmatrix}$$

b)  $AB + C^2$

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$$\begin{aligned}
 AB + C^2 &= \begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 4 & 0 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 17 & 9 \\ 19 & 21 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 0 & 9 \end{bmatrix} \\
 &= \begin{bmatrix} 21 & 14 \\ 19 & 30 \end{bmatrix}
 \end{aligned}$$

$$c) CB$$

$$CB = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 4 & 0 \\ 3 & 1 \end{bmatrix} = \text{DNE}$$

example: let  $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$

find

- a)  $AB$
- b)  $BA$

$$a) AB = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} = [23]$$

$$b) BA = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 1 & 2 & 3 \\ 6 & 12 & 18 \end{bmatrix}$$

we'll skip block multiplication

note: (can skip)

there is another way to  
multiply a matrix by a  
column vector

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 17 \\ 41 \end{bmatrix}$$