Section 3.2: Matrix Algebra

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see Properties of Matrices handout for "Algebraic Properties"
- basically, matrices <u>add</u> like you'd expect

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definition: $c_1A_1 + c_2A_2 + c_3A_3 + \dots + c_nA_n$ is a <u>linear combination</u> of the matrices $A_1, A_2, A_3, \dots A_n$

example: Is $B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$ a LC of (linear combo)

$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$?

answer: $\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} = c_1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c_2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$\begin{cases} 2 = c_2 + c_3 & \text{top left entry} \\ 1 = c_1 + c_3 & \text{top right} \\ -3 = -c_1 + c_3 & \\ 2 = c_2 + c_3 & \end{cases}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ -1 & 0 & 1 & -3 \\ 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} c_1 = 2 \\ c_2 = 3 \\ c_3 = -1 \end{array}$$

$B = 2A_1 + 3A_2 - A_3$

$\boxed{\text{yes}}$

definition: the <u>span</u> of a set of $M \times N$ matrices

definition: the <u>span</u> of a set of $M \times N$ matrices is the set of all linear combinations of the matrices

so, from our previous example,

$$B \text{ is in } \operatorname{span}(A_1, A_2, A_3)$$

definition: matrices $A_1, A_2, A_3, \ldots A_n$ are linearly independent (LI) if

$$c_1 A_1 + c_2 A_2 + \ldots c_n A_n = 0$$

only has the trivial solution

$$c_1 = c_2 = c_3 = \ldots = c_n = 0$$

- otherwise, they are linearly dependent (LD)

example: Are $A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ linearly independent?

answer: $c_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

where
$$\begin{cases} c_2 + c_3 = 0 \\ c + c_3 = 0 \\ -c_1 + c_3 = 0 \\ c_2 + c_3 = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so the solution $c_1 = c_2 = c_3 = 0$ is the only solution

A_1, A_2, and A_3 are $\boxed{LI}$

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we also looked at Properties of Matrices handout - "properties of Matrix Multiplication" section

warning: recall that in general, $BA \neq AB$ for matrices A and B

be careful!

$$(A+B)^2 = (A+B)(A+B)$$

$$= A^2 + \underbrace{AB + BA}_{} + B^2$$

okay so for

but cannot assume that $AB = BA$,

so this is generally <u>not</u> $2AB$

look at handout again, "Properties of Transpose"

the one property of the transpose that you really need to know is

$$(AB)^T = B^T A^T$$

if you think about it, this actually make a certain amount of sense

$$AB = \overset{2\times 3}{\begin{bmatrix} 2 & 3 & 7 \end{bmatrix}} \overset{3\times 1}{\begin{bmatrix} 1 \end{bmatrix}} = \overset{2\times 1}{\begin{bmatrix} \quad \end{bmatrix}}$$

$$AB = \overset{2\times 3}{\begin{bmatrix} 2 & 3 & 7 \\ 4 & 0 & 9 \end{bmatrix}} \overset{3\times 1}{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}} = \overset{2\times 1}{\begin{bmatrix} \\ \end{bmatrix}}$$

so $(AB)^T$ will be 1×2

what would $A^T B^T$ and $B^T A^T$ look like?

$$A^T B^T = \overset{3\times 2}{\begin{bmatrix} 2 & 4 \\ 3 & 0 \\ 7 & 9 \end{bmatrix}} \overset{1\times 3}{\begin{bmatrix} 1 & 1 & 4 \end{bmatrix}} = \text{not defined!}$$

$$B^T A^T = \overset{1\times 3}{\begin{bmatrix} 1 & 1 & 4 \end{bmatrix}} \overset{3\times 2}{\begin{bmatrix} 2 & 4 \\ 3 & 0 \\ 7 & 9 \end{bmatrix}} = \overset{1\times 2}{\begin{bmatrix} & \end{bmatrix}} \quad \text{works!}$$

examples:

let $A = \begin{bmatrix} 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

calculate

a) $AB = \overset{1\times 2}{\begin{bmatrix} 1 & 3 \end{bmatrix}} \overset{2\times 1}{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} = \begin{bmatrix} 14 \end{bmatrix}$

b) $BA = \overset{2\times 1}{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} \overset{1\times 2}{\begin{bmatrix} 1 & 3 \end{bmatrix}} = \begin{bmatrix} 2 & 6 \\ 4 & 12 \end{bmatrix}$

c) $A + B =$ DNE

d) $A^T + B = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

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e) $(BA)^T = \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$

f) $A^T B^T = \underset{2\times 1}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} \underset{1\times 2}{\begin{bmatrix} 2 & 4 \end{bmatrix}} = \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$

give an example of a symmetric 3x3 matrix

answer: $\begin{bmatrix} 5 & 5 & 7 \\ 5 & 0 & 13 \\ 7 & 13 & 1 \end{bmatrix}$