

Section 3.4: The LU Factorization

Monday, October 16, 2023 4:06 PM

$$\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 2 & -1 & 6 \end{bmatrix}$$

lower triangular matrix

$$\begin{bmatrix} 2 & 6 & 1 \\ 0 & 3 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

upper triangular matrix

for a linear system

$$A \vec{x} = \vec{b}$$

where A is a square matrix, we want to express A as

$$A = LU$$

where L = lower triangular matrix

U = upper triangular

note: L and U are not unique

also: textbook goes further and insists
the L be unit triangular

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}$$

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then to solve for \vec{x} :

$$A \vec{x} = \vec{b}$$

$$A\vec{x} = \vec{b}$$

$$L U \vec{x} = \vec{b}$$

$$L(U\vec{x}) = \vec{b}$$

call $U\vec{x} = \vec{y}$

$$L\vec{y} = \vec{b}$$

why is this cool?

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 13 \end{bmatrix}$$



having a lower triangular matrix
means you can substitute to
find \vec{y} easily

then do

$$U\vec{x} = \vec{y}$$

you just found this

$$\begin{bmatrix} 8 & 3 & 9 \\ 0 & 5 & 2 \\ 0 & 0 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$



solve for \vec{x}

example: Find the LU factorization of A where

$$A = \begin{bmatrix} 10 & -4 & 0 \\ -10 & 12 & -6 \\ 5 & 14 & -10 \end{bmatrix}$$

and use it to solve

$$\begin{cases} 10x - 4y = 16 \\ -10x + 12y - 6z = -2 \\ 5x + 14y - 10z = 34 \end{cases}$$

answer: Step ①: transform A into U and keep track of corresponding elementary row ops and matrices
 - use only $R_i + cR_j$ ← where R_j is above R_i
 and cR_i

but no row swaps

why these rules? then all E_i 's will be lower triangular

[note: if you need to swap rows, need permutation matrix - in textbook, but we won't cover it]

$$A = \begin{bmatrix} 10 & -4 & 0 \\ -10 & 12 & -6 \\ 5 & 14 & -10 \end{bmatrix}$$

$\downarrow R_2 + R_1$

$$\begin{bmatrix} 10 & -4 & 0 \\ 0 & 8 & -6 \\ 5 & 14 & -10 \end{bmatrix}$$

$\downarrow R_3 - \frac{5}{2}R_1$

$$\begin{bmatrix} 10 & -4 & 0 \\ 0 & 8 & -6 \\ 0 & 16 & -10 \end{bmatrix}$$

$\downarrow R_3 - 2R_2$

$$U = \begin{bmatrix} 10 & -4 & 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 10 & -9 & 0 \\ 0 & 8 & -6 \\ 0 & 0 & 2 \end{bmatrix} \quad | \quad E_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad | \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 A = U$$

$$E_1^{-1} E_2^{-1} E_3^{-1} E_3 E_2 E_1 A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

$$A = \boxed{E_1^{-1} E_2^{-1} E_3^{-1}} U$$

$$L = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$\begin{aligned} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}}_{=} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix} \end{aligned}$$

answer : $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix} \begin{bmatrix} 10 & -9 & 0 \\ 0 & 8 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

and could multiply these matrices to check that the product is A

$$b) \text{ solve } A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -2 \\ 34 \end{bmatrix}$$

answer: $A\vec{x} = \vec{b}$

$$\underbrace{L \cup}_{\vec{y}} \vec{x} = \vec{b}$$

$$\vec{y} = \vec{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 16 \\ -2 \\ 34 \end{bmatrix}$$

$$\text{so } y_1 = 16$$

$$\begin{aligned} -y_1 + y_2 &= -2 & -16 + y_2 &= -2 \quad \text{and } y_2 = 14 \\ \frac{1}{2}y_1 + 2y_2 + y_3 &= 34 & \frac{1}{2}(16) + 2(14) + y_3 &= 34 \quad \text{so } y_3 = -2 \end{aligned}$$

but remember $\vec{y} = U\vec{x}$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 & -4 & 0 \\ 0 & 8 & -6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 16 \\ 14 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 & -4 & 0 \\ 0 & 8 & -6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{aligned} \text{then } 16 &= 10x - 4y & 16 &= 10x - 4(-1) \quad \text{so } x = 2 \\ 14 &= 8y - 6z & 14 &= 8y - 6(-1) \quad y = 1 \\ -2 &= 2z & \text{so } z &= -1 \end{aligned}$$

finally $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

note: to solve 3×3 systems, Gauss-Jordan is more efficient. But for large systems, LU factorization is more efficient

happy, fun shortcut:

- you must be able to reduce A to row-echelon form (\mathcal{U}) without swapping rows

$$\rightarrow \text{only } R_i + cR_j$$

\uparrow
where j is a row
above R_i

example: find an LU factorization of

$$A = \begin{bmatrix} -3 & 1 & 2 \\ 6 & 2 & -5 \\ 9 & 5 & -6 \end{bmatrix}$$

answer: $A = \begin{bmatrix} -3 & 1 & 2 \\ 6 & 2 & -5 \\ 9 & 5 & -6 \end{bmatrix}$

$\downarrow R_2 + 2R_1$

$$\begin{bmatrix} -3 & 1 & 2 \\ 0 & 4 & -1 \\ 9 & 5 & -6 \end{bmatrix}$$

$\downarrow R_3 + 3R_1$

$$\begin{bmatrix} -3 & 1 & 2 \\ 0 & 4 & -1 \\ 0 & 7 & -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

$R_2 - 2R_1$ $R_3 - 3R_1$ $R_3 + 2R_2$

$$\left[\begin{array}{ccc} -3 & 1 & 2 \\ 0 & 4 & -1 \\ 0 & 8 & 0 \end{array} \right]$$

$$L = 3 \quad 2 \quad 1 \quad R_3 - 3R_1 \quad R_3 + 2R_2$$

$$U = \left[\begin{array}{ccc} -3 & 1 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{array} \right]$$

$\downarrow R_3 - 2R_2$

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Example: solve the system $A\vec{x} = \vec{b}$ where

$$A = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -2 & 1 \end{array} \right] \left[\begin{array}{cccc} 1 & 4 & 3 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \text{ and } \vec{b} = \left[\begin{array}{c} 1 \\ -3 \\ -1 \\ 0 \end{array} \right]$$

answer:-

$$\begin{aligned} A\vec{x} &= \vec{b} \\ U\vec{x} &= \vec{b} \\ \vec{y} & \text{ so } U\vec{y} = \vec{b} \end{aligned}$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -2 & 1 \end{array} \right] \left[\begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \right] = \left[\begin{array}{c} 1 \\ -3 \\ -1 \\ 0 \end{array} \right]$$

$$\text{so } y_1 = 1$$

$$-2y_1 + y_2 = -3 \quad \text{and} \quad -2 + y_2 = -3 \quad \text{then } y_2 = -1$$

$$3y_1 - 2y_2 + y_3 = -1 \quad \text{and} \quad 3 - 2 + y_3 = -1 \quad \text{then } y_3 = -6$$

$$-5y_1 + 4y_2 - 2y_3 + y_4 = 0 \quad \text{and} \quad -5 - 4 + 12 + y_4 = 0 \quad \text{then } y_4 = -3$$

now $U\vec{x} = \vec{y}$

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -6 \\ -3 \end{bmatrix}$$

with $x_1 + 4x_2 + 3x_3 = 1$ $x_1 - 4/3 + 9 = 1$ and $x_1 = 16/3$
 $3x_2 + 5x_3 + 2x_4 = -1$ $3x_2 + 15 - 6 = -1$ and $x_2 = -10/3$
 $-2x_3 = -6$ $so \quad x_3 = 3$
 $x_4 = -3$

so $\vec{x} = \begin{bmatrix} 16/3 \\ -10/3 \\ 3 \\ -3 \end{bmatrix}$