

Section 3.5: Subspaces, Basis, Dimension, and Rank

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definition: a subspace of \mathbb{R}^n is a collection of vectors S in \mathbb{R}^n such that

1) $\vec{0}$ (the zero vector) is in S

2) if \vec{v} and \vec{w} are in S , then $(\vec{v} + \vec{w})$ is also in S

3) if \vec{v} is in S and c is any scalar, then $c\vec{v}$ is also in S

example: in \mathbb{R}^3 , any line or plane containing the origin is a subspace

example: consider the set of vectors

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

such that $x = 2y$ and $z = -4y$. Is this a subspace of \mathbb{R}^3 ?

answer: let $y = t$ (no restrictions on y)

then $\begin{cases} x = 2t \\ y = t \\ z = -4t \end{cases}$

so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$

now look at the three conditions:

1) is $\vec{0}$ in the ~~set~~? yes, when $t = 0$

2) suppose $\begin{bmatrix} 2a \\ 1 \\ -4a \end{bmatrix}$ and $\begin{bmatrix} 2b \\ 1 \\ -4b \end{bmatrix}$ are in

2) suppose $\begin{bmatrix} 2a \\ a \\ -4a \end{bmatrix}$ and $\begin{bmatrix} 2b \\ b \\ -4b \end{bmatrix}$ are in the subspace. Is $\begin{bmatrix} 2a \\ a \\ -4a \end{bmatrix} + \begin{bmatrix} 2b \\ b \\ -4b \end{bmatrix}$ also in the subspace?

$$\begin{bmatrix} 2a \\ a \\ -4a \end{bmatrix} + \begin{bmatrix} 2b \\ b \\ -4b \end{bmatrix} = (a+b) \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

\uparrow
multiple of $\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$

so yes.

3) if $\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$ is in the subspace, then is $\begin{bmatrix} 2c \\ 1c \\ -4c \end{bmatrix}$ in the subspace?

yes, because the space is just multiples of $\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$

example: consider the set of vectors such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$x = 2y + 1 \quad \text{and} \quad z = -4y.$$

Is this set a subspace?

answer: let $\vec{y} = t$

$$\begin{cases} x = 2t + 1 \\ y = t \\ z = -4t \end{cases}$$

1) Is $\vec{0}$ in this set? No! There is no value of t that simultaneously makes all three components zero.

example: Consider the set of vectors in \mathbb{R}^2 such that $y = x^3$. Is this a subspace of \mathbb{R}^2 ?

answer: let $\vec{x} = t$

$$\begin{cases} x = t \\ y = t^3 \end{cases}$$

1) Is $\vec{0}$ in the set? Yes, when $t=0$

2) Suppose $\begin{bmatrix} a \\ a^3 \end{bmatrix}$ and $\begin{bmatrix} b \\ b^3 \end{bmatrix}$ are in the

set. Is $\begin{bmatrix} a \\ a^3 \end{bmatrix} + \begin{bmatrix} b \\ b^3 \end{bmatrix}$ in the set?

$$\begin{bmatrix} a+b \\ a^3 + b^3 \end{bmatrix} \xleftarrow{\text{sum}} \text{but } \begin{bmatrix} a+b \\ (a+b)^3 \end{bmatrix} \text{ is}$$

what you get from

$$\begin{cases} x = t \\ y = t^3 \end{cases}$$

does $a^3 + b^3 = (a+b)^3$

No, set is not a subspace

Given any matrix A , there are three fundamental spaces associated with it.

- (1) the row space
- (2) the column space
- (3) the null space

example : let $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 4 & 3 \end{bmatrix}$

what is $\text{Row}(A)$? $\text{Col}(A)$?

answers: $\text{Row}(A)$ = set of all vectors we can make from linear combinations of the rows of A

$$= \text{span}([1, -2], [0, 1], [4, 3])$$

- this is a subspace of \mathbb{R}^2

(in fact, in this particular example the subspace is \mathbb{R}^2)

$$\text{Col}(A) = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right)$$

this is a subspace of \mathbb{R}^3
and is a plane through
the origin

definition : let A be an $M \times N$ matrix

- ① the row space of A is a subspace of \mathbb{R}^N spanned by the rows of A . It is denoted by $\text{Row}(A)$.
- ② the column space of A is a subspace of \mathbb{R}^m spanned by the columns of A . It is denoted by $\text{Col}(A)$.

example: let $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 2 & -4 \end{bmatrix}$

a) Is $\vec{b} = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$ in $\text{Col}(A)$?

answer:

- is \vec{b} in the span of these vectors (columns)?
- is \vec{b} a linear combination of these columns?

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -1 \\ 2 & -4 & 8 \end{array} \right] \xrightarrow{\text{RREF}}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right]$$

$$\text{so } \vec{b} = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix}$$

yes

b) Is $\vec{w} = [3 \ 2]^T$ in $\text{Row}(A)$?

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

\sim is $w = L \circ d$... more.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -4 \end{bmatrix}$$

method #1: (not recommended)

you could, if you insist, solve

$$\left[\begin{array}{cc|c} 1 & -2 & \\ 0 & 1 & \\ 2 & -4 & \\ \hline 3 & 2 & \end{array} \right]$$

this augmented matrix
needs column
operations (not row ops)
to get RREF
 \uparrow
column

method #2: take the transpose

$$[A^T | \vec{w}^T]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ -2 & 1 & -4 & 2 \end{array} \right] \\ \downarrow R_2 + 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 8 \end{array} \right]$$

free variable, infinitely many
solutions

yes

note: if your RREF was

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

answer is no

theorem: if two matrices are row-equivalent,
then they have the same row space

so if

$$A \xrightarrow{\text{row ops}} B$$

$$\text{then } \text{Row}(A) = \text{Row}(B)$$

example: consider $A = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}$ $\xrightarrow{\text{RREF}}$ $\begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$

a) find $\text{Row}(A)$.

$$\text{Row}(A) = \text{span}(\begin{bmatrix} 1, 5 \end{bmatrix}, \begin{bmatrix} 2, 10 \end{bmatrix})$$

$$= \text{span}(\begin{bmatrix} 1, 5 \end{bmatrix})$$

↑ because this is a multiple of the first

b) find $\text{Col}(A)$

$$\text{Col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 10 \end{bmatrix}\right)$$

$$= \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

note: cannot read this straight off the RREF

$$\text{RREF} = \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\underline{\text{so}}} \quad \text{Row}(A) = \text{Row}(\text{RREF})$$

$$\text{Col}(A) \neq \text{Col}(\text{RREF})$$

conclusion: row operations preserve the row space
but change the column space

Section 3.5: contd:

definition : a **basis** of a subspace S in \mathbb{R}^n is a set of vectors in S such that

- 1) the span of the vectors in the basis is S
- 2) the set of vectors in the basis is LI

note: I like to think of a basis as the minimum set of vectors needed to span the space

example: the set of vectors $\{\hat{i}, \hat{j}, \hat{k}\}$ is a basis of \mathbb{R}^3

why do we care? for every vector \vec{v} in subspace S there is exactly one way to write \vec{v} as a linear combo of the basis vectors for any basis

notation :

$\{\}$ is used for a set

$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ is a set of vectors

a set is an unordered collection of objects
(numbers, matrices, vectors)

$\text{span}(\)$

$\underbrace{\quad}_{\text{function notation}} f(x)$

definition: the dimension of a subspace S , where S is not the zero vector, is the number of vectors in a basis for S

if S is the zero vector, then the dimension of S is zero

examples: the subspaces of \mathbb{R}^3

1) the point $\vec{0}$ has dimension 0

2) any line through the origin has dimension 1

3) any plane through the origin has dimension 2

4) the entire space \mathbb{R}^3 is a volume with dimension 3

definition: $\text{Null}(A) = \left\{ \vec{x} \in \mathbb{R}^N \mid A\vec{x} = \vec{0} \right\}$

↑
the set of all
vectors \vec{x}
in \mathbb{R}^N

"such
that"

↑
this equation
holds

the null space of an $M \times N$ matrix A is the set of all vectors in \mathbb{R}^N such that when those vectors are left-multiplied by A , the result is the zero vector $\vec{0}$

note: row ops on A do not change the null space

example: find a basis for the row space, the column space, and the null space of

$$A = \begin{bmatrix} 3 & 7 & 4 & 2 & 0 \\ 4 & 3 & -1 & 3 & 1 \\ 9 & 8 & 9 & 5 & 7 \\ 5 & 4 & -1 & 6 & 8 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

answer: row space

$$\begin{aligned} \text{Row}(A) &= \text{Row}(\text{RREF}) \\ &= \text{span}(\text{non-zero rows of RREF}) \end{aligned}$$

so a basis for $\text{Row}(A)$ is

$$\left\{ \begin{bmatrix} 1 & 0 & -1 & 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 3 \end{bmatrix} \right\}$$

↑ →
the curly brackets mean
the set of these three vectors

column space - recall that row operations change the column space, so we can't just read off the RREF

- but the RREF will tell us which in the original matrix to use

$$\text{Col}(A) = \text{span}(\text{columns of } A \text{ with leading ones in the RREF})$$

there are leading ones in columns
1, 2, and 4

so basis of $\text{Col}(A)$ is

$$\left\{ \begin{bmatrix} 3 \\ 4 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \\ 6 \end{bmatrix} \right\}$$

note: column 3 and column 5 were not included because they are linear combos of earlier columns

- the third column of the RREF
is a recipe for column 3

$$\vec{C}_3 = -1 \vec{C}_1 + 1 \vec{C}_2$$

similarly, $\vec{C}_5 = -2 \vec{C}_1 + 3 \vec{C}_2$

null space: $\text{Null}(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$
the set of all \vec{x} such that $A\vec{x} = \vec{0}$

$$A\vec{x} = \begin{bmatrix} 3 & 7 & 4 & 2 & 0 \\ 4 & 3 & -1 & 3 & 1 \\ 4 & 8 & 4 & 5 & 7 \\ 5 & 4 & -1 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$4 \times 5 \qquad \qquad \qquad 5 \times 1$

$$[\vec{A} \mid \vec{0}] \xrightarrow{\text{row ops}} [\text{RREF } \mid \vec{0}]$$

row ops do not change null space

row ops do not change null space

$$\left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & 0 \\ 1 & 0 & -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$\uparrow \quad \uparrow$

free variables

assign parameters to free variables:

let $x_3 = s$
 $x_5 = t$

$$\left\{ \begin{array}{l} x_1 = -x_3 - 2x_5 = 0 \\ x_2 = +x_3 = 0 \\ x_4 = +3x_5 = 0 \end{array} \right.$$

sub in parameters

$$\left\{ \begin{array}{l} x_1 = s + 2t \\ x_2 = -s \\ x_3 = s \\ x_4 = -3t \\ x_5 = t \end{array} \right.$$

$$\vec{x} = s \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

so a basis for the null space is

$$\left\{ \left[\begin{array}{c} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{array} \right], \left[\begin{array}{c} 2 \\ 0 \\ 0 \\ -3 \\ 1 \end{array} \right] \right\}$$

quick conceptual example:

there is a light source directly overhead that will cast the shadow of your fingertip onto a desk.

let $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the position of your fingertip in space

then $\vec{y} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ is the position of your fingertip's shadow on the desk

and the projection onto the desk (position of shadow) is

$$\vec{y} = A \vec{x}$$

where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

*note: this is
in RREF
already*

question: give a basis for the row space and null space of A

answer: row space: $\{ [1 \ 0 \ 0], [0 \ 1 \ 0] \}$

so row space is xy-plane,
where moving your finger in
this plane makes the

where moving your finger in this plane moves the shadow

null space:

$$A\vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

↑ free variable, assign parameter
let $z = t$

$$\begin{cases} x = 0 \\ y = 0 \\ z = t \end{cases}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

↑ basis for null space

moving your fingertip along z -axis does not move the shadow

in general,

$$\text{Rank}(A) = \text{dimension of Row}(A)$$

- number of vectors in row space

$$= \text{dimension of Col}(A)$$

- number of vectors in column space

$$= \text{number of leading ones}$$

in RREF

$$\text{Nullity}(A) = \text{dimension of Null}(A)$$

- number of vectors in null space

= number of free variables
in RREF

recall: every column in RREF has either
a leading variable or a free variable

Theorem (Rank - Nullity Theorem)

if matrix A is MxN

then N is the number of variables

$$\begin{aligned} N &= \# \text{leading} + \# \text{free} \\ &= \text{Rank}(A) + \text{Nullity}(A) \end{aligned}$$

example: let $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Find $\text{rank}(A)$ and $\text{nullity}(A)$.

answer: $\text{rank}(A) = 3$
 $\text{nullity}(A) = 1$

example: Consider subspace S, where

$$S = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$$

where

$$\begin{aligned} \vec{v}_1 &= \begin{bmatrix} 1 & -2 & -3 & 2 & -4 \end{bmatrix} \\ \vec{v}_2 &= \begin{bmatrix} -3 & 7 & -1 & 1 & -3 \end{bmatrix} \\ \vec{v}_3 &= \begin{bmatrix} 2 & -5 & 4 & 3 & 7 \end{bmatrix} \\ \vec{v}_4 &= \begin{bmatrix} -3 & 6 & 9 & -6 & 1 \end{bmatrix} \end{aligned}$$

* note: these
are all
row vectors

$$\vec{v}_3 = \begin{bmatrix} 2 & -5 & 4 & 3 & 7 \end{bmatrix}$$

$$\vec{v}_4 = \begin{bmatrix} -3 & 6 & 9 & -6 & 1 \end{bmatrix}$$

row vectors

note: $A = \begin{bmatrix} 1 & -2 & -3 & 2 & -4 \\ -3 & 7 & -1 & 1 & -3 \\ 2 & -5 & 4 & 3 & 7 \\ -3 & 6 & 9 & -6 & 1 \end{bmatrix}$ has RREF

$$\begin{bmatrix} 1 & 0 & -23 & 16 & 0 \\ 0 & 1 & -10 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- a) find a basis for $S \leftarrow$ want row space
- b) find a basis for S consisting of the original vectors

answer: a) $S = \text{Row}(A)$ and a basis for $\text{Row}(A)$
 is the set of non-zero rows in the
 RREF of A

$$\text{basis} = \left\{ \begin{bmatrix} 1 & 0 & -23 & 16 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -10 & 7 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

Section 3.5: cont'd 2023/10/25

- b) What we want is to select 3 of the 4 original rows, but we don't know which ones to pick

- but the method to find $\text{Col}(A)$ gives the original columns in the answer so use transpose

$$A^T = \begin{bmatrix} \vec{v}_1^T & \vec{v}_2^T & \vec{v}_3^T & \vec{v}_4^T \\ 1 & -3 & 2 & -3 \\ -2 & 7 & -5 & 6 \\ -3 & -1 & 4 & 9 \\ 2 & 1 & -3 & -6 \\ -4 & -3 & 7 & 1 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$S = \text{Col}(A^T)$
(ess)

The basis we want is $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$

Section 3.5: cont'd 2023/10/30

Theorem: Let S be a subspace of \mathbb{R}^n and let $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be a basis for S .

Then any vector \vec{v} in S can be uniquely expressed as a linear combination:

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$$

Notation

$$[\vec{v}]_B =$$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}_B$$

These are the coordinates of \vec{v} with respect to B , but also the coeffs of the linear combo

if there isn't any subscript, then the default in \mathbb{R}^3 would be

$$\{\hat{i}, \hat{j}, \hat{k}\}$$

example: let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

a) show that $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of \mathbb{R}^3

b) find the coordinates of $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ with respect to B .

answer: a) a basis of \mathbb{R}^3 requires 3 LI vectors

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$\uparrow \uparrow \uparrow$
no free variables

\therefore LI

and B is a basis for \mathbb{R}^3

b) want coords of \vec{v} with respect to B

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 0 & 1 & 3 \\ 0 & 3 & 1 & 4 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 13 \end{array} \right]$$

$$\text{so } \vec{v} = -5\vec{v}_1 - 3\vec{v}_2 + 13\vec{v}_3$$

$$\vec{v} = \begin{bmatrix} -5 \\ -3 \\ 13 \end{bmatrix}_B$$