

Section 4.1: Intro to Eigenvectors and Eigenvalues

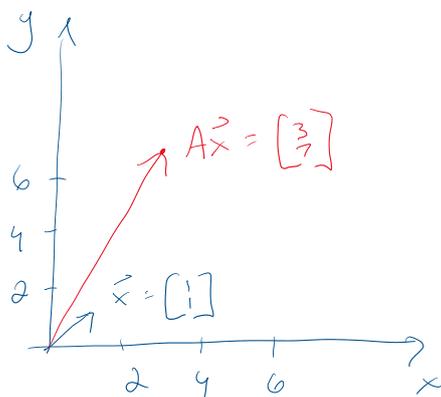
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let A be a 2×2 matrix

typically, $A\vec{x}$ scales and rotates \vec{x}

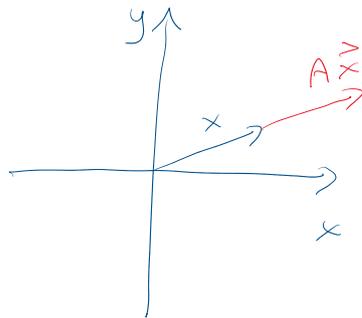
example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

then $A\vec{x} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$



\vec{x} has been rotated and scaled when multiplied by A

we are interested in the special vectors that are simply scaled by A and not rotated



so $A\vec{x}$ is a multiple of \vec{x}

definition: let A be an $N \times N$ matrix

a scalar λ is called an eigenvalue
"lambda"

if there is a non-zero vector \vec{x} such that

$$A\vec{x} = \lambda\vec{x}$$

\uparrow \uparrow
 A is a λ is a constant
 matrix

and we also say that \vec{x} is the eigenvector corresponding to λ

example: $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. Show that $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors of A and find their associated eigenvalues.

answer: $A\vec{x}_1 = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\vec{x}_1$

$\lambda_1 = 2$

$$A\vec{x}_2 = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\vec{x}_2$$

$\lambda_2 = 3$

but how can you find λ if you don't know \vec{x} ?

$$A\vec{x} = \lambda\vec{x}$$

$$A\vec{x} - \lambda\vec{x} = \mathbf{0}$$

$$A\vec{x} - \lambda I\vec{x} = \mathbf{0}$$

$$(A - \lambda I)\vec{x} = \mathbf{0}$$

$$\begin{bmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

we don't want to be able to find a unique solution and reduce this $(A - \lambda I)$ to the identity matrix because the only solution will be

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and we are interested in non-zero vectors

so we want $A - \lambda I$ to not have an inverse - we want the determinant to equal zero

$$\det(A - \lambda I) = 0$$

this gives us a polynomial in λ (called the characteristic equation) whose roots are the eigenvalues of A

example: Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

answer:

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(4-\lambda) - 2(-1) = 0$$

$$4 - 5\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 3)(\lambda - 2) = 0$$

Section 9.1: cont'd

$$\lambda = 3, 2$$

note: a 2×2 matrix can have

1) 2 distinct real eigenvalues

2) 1 real repeated eigenvalue
(characteristic polynomial was a perfect square)

3) 2 complex eigenvalues

In general, for an $N \times N$ matrix A , we can find its eigenvalues by solving

$$\det(A - \lambda I_N) = 0$$

and the associated eigenvectors are found by solving

$$(A - \lambda I_N) \vec{x} = \vec{0}$$

example: now find the eigenvectors for the eigenvalues in the previous example.

note: we found that $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

has eigenvalues

$$\lambda_1 = 2, \lambda_2 = 3$$

answer: now solve $(A - \lambda I) \vec{x} = \vec{0}$

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & 2 \\ -1 & 4 - \lambda \end{bmatrix}$$

so use $\lambda_1 = 2$ $(A - \lambda I) \vec{x} = \vec{0}$

$$\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

↑

$$\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

y is free variable
let $y = t$

$$\begin{cases} x = 2t \\ y = t \end{cases}$$

$$\vec{x}_1 = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

so $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector
as is $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ is an eigenvector,
and all multiples of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
will also be eigenvectors

optional check:

$$A \vec{x}_1 = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \vec{x}_1$$

so now do $\lambda_2 = 3$:

$$\text{solve } (A - \lambda I) \vec{x} = \vec{0}$$

$$\begin{bmatrix} 1 - \lambda & 2 \\ -1 & 4 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}$$

RREF \rightarrow

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

let $y = t$ (free variable)

$$\begin{cases} x = t \\ y = t \end{cases}$$

\Rightarrow . . .

$$\vec{x}_2 = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

final answer: $\lambda_1 = 2$ has eigenvector $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
 $\lambda_2 = 3$ has eigenvector $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

definition: eigenspace - the collection of all eigenvectors corresponding to a certain λ is called an eigenspace and is denoted by E_λ

so for previous example,

$$\text{for } \lambda_1 = 2, \quad E_{\lambda_1} = \text{span} \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$$

$$\lambda_2 = 3, \quad E_{\lambda_2} = \text{span} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

note: observe that if \vec{x} is an eigenvector for a certain λ , then any multiple of \vec{x} is also an eigenvector for that λ

also, if \vec{x}_1 and \vec{x}_2 are both eigenvectors corresponding to the same λ , then the vector $(\vec{x}_1 + \vec{x}_2)$ is also an eigenvector