

Section 4.2: Determinants

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2×2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3×3 matrices:

we can calculate the determinant in the same way we did the cross product:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 5 \\ 3 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 2(0)(2) + 1(5)(-1) + 3(4)(-1) \\ &\quad - 3(0)(3) - 2(5)(2) - 1(4)(-1) \\ &= 0 + 15 - 12 - 0 + 10 - 8 \\ &= 5 \qquad \leftarrow \text{the answer is a single number} \end{aligned}$$

The method of minors:

general case of how to calculate a determinant

start with a 3×3 :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

positive in upper left corner

first, assign the signs:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} - & + & - \\ + & - & + \end{bmatrix}$$

alternate everywhere
else

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$$\text{then } \det(A) = +a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

here, I've expanded about the first row, but you can do this about any row or column

example: Find the determinant of

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 5 \\ 3 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

using the method of minors.

a) across Row 1:

$$\det(A) = +2 \begin{vmatrix} 0 & 5 \\ -1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} + 3 \begin{vmatrix} 4 & 0 \\ 3 & -1 \end{vmatrix}$$

$$= 2(0+5) - 1(8-15) + 3(-4-0)$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 5 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} = 10 + 7 - 12 = 5$$

b) across Row 2:

$$\det(A) = -4 \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} + 0 \begin{vmatrix} \text{don't} \\ \text{care} \end{vmatrix} - 5 \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= -4(2+3) + 0 - 5(-2-3)$$

$$= 5$$

note: could expand across Column 2 instead

example: Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & 0 & 6 & 0 \\ 2 & 1 & 5 & 6 \\ 3 & 9 & -2 & 7 \end{bmatrix}$$

$\begin{bmatrix} + & & & \\ - & + & - & + \end{bmatrix}$

answer: $\det(A) = -0 + 0 - 6 \begin{vmatrix} 1 & 2 & 9 & | & 1 & 2 \\ 2 & 1 & 6 & | & 2 & 1 \\ 3 & 9 & 7 & | & 3 & 4 \end{vmatrix}$

$= -6(7 + 36 + 32 - 12 - 24 - 28)$

$= -6(11) = -66$

special case: triangular matrices

example: find determinant of $A = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 0 & 5 & 1 & 6 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

answer: expand about column 1

$$\begin{aligned} \det(A) &= +2 \begin{vmatrix} 5 & 1 & 4 & | & 1 \\ 0 & 5 & 1 & | & 4 \\ 0 & 0 & 4 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{vmatrix} \\ &= 2(5)(4)(2) \quad \text{because of all the zeros} \\ &= 80 \end{aligned}$$

In general, the determinant of an upper or lower triangular matrix is the product of the main diagonal

Properties of Determinants

① $\det(A^T) = \det(A)$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\det(A) = 1(4) - 2(3)$$

② If A has a row or column with all zeros, then $\det(A) = 0$.

③ If $A \rightarrow B$ (row swap operation)
 $R_i \leftrightarrow R_j$

then $\det(A) = -\det(B)$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \det(A) = -2$$

$$B = \begin{bmatrix} 3 & 9 \\ 1 & 2 \end{bmatrix} \quad \det(B) = 2$$

④ If $A \xrightarrow{kR_i} B$ (multiply row by constant)

then $k \det(A) = \det(C)^*$

$$C = \begin{bmatrix} 5 & 10 \\ 3 & 4 \end{bmatrix} \in \text{mult Row 1 by } 5$$

* also true if you multiply a column by k \det(C) = -10

⑤ If $A \xrightarrow{R_i + kR_j} B$ (add multiple of another row)

then $\det(A) = \det(B)$

⑥ If A has two identical rows or columns
 then $\det(A) = 0$

example: Find determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 5 & 1 \end{bmatrix}$

$$\downarrow R_2 - 2R_1$$

$\downarrow R_2 - 2R_1$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

$\det(B) = 0$ but $R_2 - 2R_1$ doesn't change
the determinant so $\det(A) = 0$

further properties of determinants:

if A and B are $N \times N$ matrices and k is a scalar, then

$$\textcircled{1} \quad \det(AB) = \det(A) \det(B)$$

$$\textcircled{2} \quad \det(kA) = k^N \det(A) \quad \leftarrow \text{know this one}$$

$$\textcircled{3} \quad \det(A^{-1}) = \frac{1}{\det(A)} \quad \text{why?}$$

$$\det(AA^{-1}) = \det(I)$$

$$\det(A)\det(A^{-1}) = 1$$

optional: inverse of a matrix: cofactor method

example: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$

using the cofactor method.

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answer: $C = \begin{bmatrix} + \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} & - \begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} & + \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} \\ - \begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} & + \begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} & - \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} \\ + \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} & - \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} & + \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{bmatrix}$

$$C = \begin{bmatrix} 1 & -10 & 7 \\ 1 & 4 & -3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{bmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} C^T \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{bmatrix} \end{aligned}$$

application of determinants: Cramer's Rule

notation: If $A\vec{x} = \vec{b}$, then $A_i(\vec{b})$ is the matrix in which column i is replaced by \vec{b} .

Cramer's Rule: If A is $N \times N$ and $A\vec{x} = \vec{b}$, then

$$x_i = \frac{\det(A_i(\vec{b}))}{\det(A)}$$

example: For the following system, solve for y using Cramer's Rule.

$$\begin{cases} x + 2y - z = 2 \\ 3x + 7y - 5z = 5 \\ -x - 2y = 1 \end{cases}$$

answer:

$$\left[\begin{array}{ccc|c} & A & & \vec{b} \\ \begin{matrix} 1 & 2 & -1 \\ 3 & 7 & -5 \\ -1 & -2 & 0 \end{matrix} & \left[\begin{matrix} \vec{x} \\ \vec{y} \\ \vec{z} \end{matrix} \right] & = & \left[\begin{matrix} 2 \\ 5 \\ 1 \end{matrix} \right] \end{array} \right]$$

$$A = \left[\begin{array}{ccc} 1 & 2 & -1 \\ 3 & 7 & -5 \\ -1 & -2 & 0 \end{array} \right] \quad \det(A) = -1$$

$$A_2(\vec{b}) = \left[\begin{array}{ccc} 1 & 2 & -1 \\ 3 & 5 & -5 \\ -1 & 1 & 0 \end{array} \right] \quad \det(A_2(\vec{b})) = 7$$

$$y = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{7}{-1} = -7$$