Section 4.3: Eigenvalues and Eigenvectors

Friday, November 10, 2023 1:23 PM

3×3 case:

example: let $A = \begin{bmatrix} -4 & 0 & 3 \\ 0 & 2 & 0 \\ -6 & 0 & 5 \end{bmatrix}$

Find all eigenspaces of A.

answer: step ①: find eigenvalues λ using $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \left|\begin{array}{ccc|cc} -4-\lambda & 0 & 3 & -4-\lambda & 0 \\ 0 & 2-\lambda & 0 & 0 & 2-\lambda \\ -6 & 0 & 5-\lambda & -6 & 0 \end{array}\right.$$

$$0 = (-4-\lambda)(2-\lambda)(5-\lambda) + 18(2-\lambda)$$

$$0 = (2-\lambda)\left[(-4-\lambda)(5-\lambda) + 18\right]$$

$$0 = (2-\lambda)\left[-20 - \lambda + \lambda^2 + 18\right]$$

$$0 = (2-\lambda)(\lambda^2 - \lambda - 2)$$

$$0 = (2-\lambda)(\lambda - 2)(\lambda + 1)$$

$$\lambda = \underbrace{2, 2}_{\text{multiplicity } 2}, \underbrace{-1}_{\text{multiplicity } 1}$$

multiplicity = number of times the value occurs in the solution

step ②: find eigenvectors using $(A - \lambda I)\vec{x} = 0$

for $\lambda_1 = 2$: $(A - \lambda I)\vec{x} = 0$

$$\begin{bmatrix} -4-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ -6 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -6 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & 0 & 3 & 0 \end{array}\right]$$

$\Downarrow$ RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad x - \tfrac{1}{2}z = 0$$

$\uparrow$ $y = s$ $\quad$ $\uparrow$ $z = t$ $\quad$ free variables

$$\begin{cases} x = \frac{1}{2}t \\ y = s \\ z = t \end{cases} \qquad \text{so that} \qquad \vec{x} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

now do $\lambda_2 = -1$: $\quad$ solve $(A - \lambda_2 I)\vec{x} = 0$

$$\left[\begin{array}{ccc|c} -4-\lambda_2 & 0 & 3 & 0 \\ 0 & 2-\lambda_2 & 0 & 0 \\ -6 & 0 & 5-\lambda_2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ -6 & 0 & 6 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x - z &= 0 \\ y &= 0 \end{aligned}$$

$\uparrow$ let $z = t$

$$\begin{cases} x = t \\ y = 0 \\ z = t \end{cases} \qquad \text{so then} \qquad \vec{x} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

step 3: write out eigenspaces

for $\lambda_1 = 2$, $E_{\lambda_1} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}\right)$

$\lambda_2 = -1$, $E_{\lambda_2} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

The <u>algebraic</u> multiplicity of an eigenvalue is the number of times the eigenvalue is repeated as a root of

$$\det(A - \lambda I) = 0$$

example: if $\det(A - \lambda I) = (\lambda + 1)^3 (\lambda - 2)(\lambda - 4)^2$

then

eigenvalue	algebraic mult
-1	3
2	1
4	2

The geometric multiplicity of an eigenvalue is the dimension of E_λ, (the number of vectors in the basis of the eigenspace)

- in our previous 3×3 example, we found that for

$\lambda_1 = 2$, $E_{\lambda_1} = \text{span}\left(\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

so $\dim(E_{\lambda_1}) = 2$ (2D space - plane)

so $\lambda_1 = 2$ has geometric multiplicity of 2

- the algebraic and geometric multiplicity of an eigenvalue can be different, but

geo mult $\leq$ algebraic mult

(if geo < alg, we say that the matrix is defective)

example: Find all eigenvalues and eigenvectors for the following matrix A. Then give the algebraic and geometric multiplicities for each eigenvalue.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

answer: $\det(A - \lambda I) = 0 = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix}$

$$= (1-\lambda)^2(2-\lambda)$$

$$\lambda = 1, \quad 2$$

algebraic mult = 2   algebraic mult = 1

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so $\lambda_1 = 1$ solve $(A - \lambda_1 I)\vec{x} = 0$

$$\left[\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} y = 0 \\ z = 0 \end{array}$$

free variable let $x = t$

$$\begin{cases} x = t \\ y = 0 \\ z = 0 \end{cases} \quad \text{and} \quad \vec{x}_1 = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

geometric mult of λ_1 is 1

now $\lambda_2 = 2$

$$(A - \lambda I)\vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x - 5z = 0 \\ y - 3z = 0 \end{array}$$

let $z = t$

$$\begin{cases} x = 5t \\ y = 3t \\ z = t \end{cases} \quad \text{and} \quad \vec{x}_2 = t\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

final answer:

eigenvalue	eigenvector(s)	algebraic mult	geometric mult
1	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	2	1
2	$\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$	1	1

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for $A\vec{x} = \lambda\vec{x}$:

$$A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda^2\vec{x}$$
$$A^3\vec{x} = A(A^2\vec{x}) = A(\lambda^2\vec{x}) = \lambda^2(A\vec{x}) = \lambda^3\vec{x}$$

and $A^N\vec{x} = \lambda^N\vec{x}$ for any $N \geq 1$

example: given $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$, find $A^{100}\vec{v}$.

answer: find eigenvalues and eigenvectors of A

note: we previously found that for this A,

eigenvalue	eigenvector
$\lambda_1 = 2$	$\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

eigenvalue	eigenvector
$\lambda_1 = 2$	$\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
$\lambda_2 = 3$	$\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and we can see that $\vec{x}_1$ and $\vec{x}_2$ are LI so that the set $\{\vec{x}_1, \vec{x}_2\}$ forms a basis for $\mathbb{R}^2$

useful fact: eigenvectors obtained from distinct eigenvalues are <u>always</u> LI

$$\begin{bmatrix} 7 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 7 \\ 1 & 1 & 3 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \end{array}\right]$$

$$\vec{v} = 4\vec{x}_1 - \vec{x}_2$$

now sub into $A^{100}\vec{v}$:

$$A^{100}\vec{v} = A^{100}(4\vec{x}_1 - \vec{x}_2)$$

$$= 4A^{100}\vec{x}_1 - A^{100}\vec{x}_2$$

$$= 4\lambda_1^{100}\vec{x}_1 - \lambda_2^{100}\vec{x}_2$$

$$= 4 \cdot 2^{100}\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 3^{100}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} (2^{103} - 3^{100}) \\ (2^{102} - 3^{100}) \end{bmatrix}}_{\text{this matrix is } 2\times 1}$$

single number

<u>complex eigenvalues:</u>

example: find the eigenvalues for

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

answer: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-1-\lambda) + 8 = 0$$

$$-3 + \lambda - 3\lambda + \lambda^2 + 8 = 0$$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2} = 1 \pm 2i$$

note: if matrix A has real entries, complex eigenvalues always occur in conjugate pairs

now let's find eigenvectors

$\lambda_1 = 1 + 2i$ $\qquad A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

solve $(A - \lambda I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 2-2i & -2 & 0 \\ 4 & -2-2i & 0 \end{array}\right]$$

method #1: $\downarrow \frac{1}{2-2i} R_1$ $\qquad$ * could multiply by $(2+2i)R_1$ instead

$$\left[\begin{array}{cc|c} 1 & \frac{-2}{2-2i} & 0 \\ 4 & -2-2i & 0 \end{array}\right]$$

but what is $\frac{-2}{2-2i}$?

$$\frac{-2}{2-2i}\left(\frac{2+2i}{2+2i}\right) = \frac{-4-4i}{4-4i^2} = \frac{-4-4i}{8} = -\tfrac{1}{2}-\tfrac{1}{2}i$$

augmented matrix is:

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2}-\frac{1}{2}i & 0 \\ 4 & -2-2i & 0 \end{array}\right]$$

$$\downarrow R_2 - 4R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2}-\frac{1}{2}i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

method #2 for finding the RREF (please use this one!)

$$\left[\begin{array}{cc|c} 2-2i & -2 & 0 \\ 4 & -2-2i & 0 \end{array}\right]$$

$$\downarrow R_1 \Leftrightarrow R_2$$

$$\left[\begin{array}{cc|c} 4 & -2-2i & 0 \\ 2-2i & -2 & 0 \end{array}\right]$$

$$\downarrow \tfrac{1}{4}R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2}-\frac{1}{2}i & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x + \left(-\tfrac{1}{2}-\tfrac{1}{2}i\right)y = 0$$

if we have done this correctly, the second row of a 2x2 here will always be a multiple of first

free variable so $y = t$

now find the eigenvector

$$\begin{cases} x = \left(\frac{1}{2}+\frac{1}{2}i\right)t \\ y = t \end{cases} \qquad \vec{x}_1 = t\begin{bmatrix} \frac{1}{2}+\frac{1}{2}i \\ 1 \end{bmatrix}$$

$$= t\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$ if you prefer

Section 4.3 : cont'd

notice we have only found

$$\lambda_1 = 1 + 2i \quad \text{has} \quad \vec{x}_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

but we will find that the complex conjugate of λ_1 will have the complex conjugate of $\vec{x}_1$ as the eigenvector

$$\text{so} \quad \lambda_2 = 1 - 2i \quad \text{has} \quad \vec{x}_2 = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$

and if you want to be really terse, you could also write

$$\text{for} \quad \lambda = 1 \pm 2i \quad , \quad \text{then} \quad \vec{x} = \begin{bmatrix} 1 \pm i \\ 2 \end{bmatrix}$$

in webwork, they go one step further and write

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \pm i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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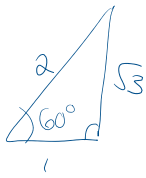
example : find the eigenvalues of $R(60°)$

answer: $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



$$R(60°) = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$\det(A - \lambda I) = 0 = \begin{vmatrix} 1/2 - \lambda & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 - \lambda \end{vmatrix}$$

$$0 = (1/2 - \lambda)(1/2 - \lambda) + 3/4$$

$$(1/2 - \lambda)^2 = -3/4$$

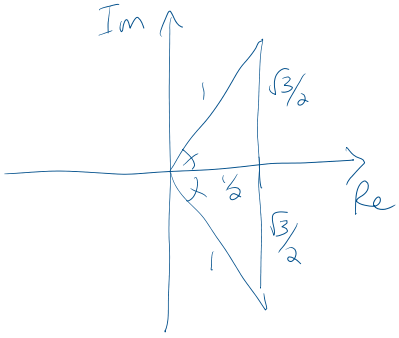
$$\left(\tfrac{1}{2}-\lambda\right)^2 = -3/4$$

$$\tfrac{1}{2}-\lambda = \pm\sqrt{-3/4}$$

$$-\lambda = -\tfrac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\lambda = \tfrac{1}{2} \pm i\sqrt{3}/2 \qquad \text{or} \qquad \frac{1 \pm i\sqrt{3}}{2}$$

but in the complex plane, what do these eigenvalues look like?



the angles are 60°

and the eigenvalues are

$$1e^{\pm i\pi/3}$$

in general, the eigenvalues of R_θ, the rotation matrix, are

$$e^{i\theta} \text{ and } e^{-i\theta}$$