Section 4.4: Diagonalization

Wednesday, November 15, 2023 9:58 AM

note: we omit similarity

definition: a square matrix A is diagonalizable if there is a diagonal matrix D and an invertible matrix P such that

$$P^{-1} A P = D$$

or

$$A = P D P^{-1}$$

example: we found earlier that matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

has $\lambda_1 = 2$ with $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$ with $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

then $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

or $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

λ_2 $\vec{x}_2$

check: $A = P D P^{-1}$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{1} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \checkmark$$

theorem: an $N \times N$ matrix A is diagonalizable if A has N linearly independent eigenvectors

example: a) a certain 3×3 matrix has 2 eigenvalues (one is repeated), but the repeated value has two eigenvectors

- 3 eigenvectors in total

$\rightarrow$ diagonalizable

b) a different 3×3 matrix has 2 eigenvalues (one is repeated), and the repeated value has only one eigenvector

- 2 eigenvectors in total

$\rightarrow$ not diagonalizable

example: are the following matrices diagonalizable? show your reasoning.

a) $A = \begin{bmatrix} 1 & -7 & 3 \\ -1 & -1 & 1 \\ 4 & -4 & 0 \end{bmatrix}$

answer: find eigenvalues by $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & -7 & 3 \\ -1 & -1-\lambda & 1 \\ 4 & -4 & -\lambda \end{vmatrix} \begin{matrix} 1-\lambda & -7 \\ -1 & -1-\lambda \\ 4 & -4 \end{matrix} = 0$$

$$(1-\lambda)(-1-\lambda)(-\lambda) - 28 + 12 - 3(4)(-1-\lambda) + 7\lambda + 4(1-\lambda) = 0$$

$$(1-\lambda)(1+\lambda)\lambda - 16 + 12(1+\lambda) + 7\lambda + 4(1-\lambda) = 0$$

$$\lambda - \lambda^3 - 16 + 12 + 12\lambda + 7\lambda + 4 - 4\lambda = 0$$

$$\lambda - \lambda^3 - 16 + 12 + 12\lambda + 7\lambda + 4 - 4\lambda = 0$$

$$16\lambda - \lambda^3 = 0$$

Section 4.4: cont'd 2023/11/17

$$\lambda(4-\lambda)(4+\lambda) = 0$$

$$\lambda = 0, 4, -4$$

3 distinct eigenvalues means 3 linearly independent eigenvectors, so

$\boxed{A \text{ is diagonalizable}}$

b) $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

answer: find eigenvalues from $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

characteristic equation $\rightarrow$ $(1-\lambda)^2(2-\lambda) = 0$

$\lambda = 1, 2$

alg mult 2 alg mult 1

we know that $\lambda_2 = 2$ has alg. mult. 1 so will have one eigenvector but

$\lambda_1 = 1$ with alg mult 2 could have either one or two eigenvectors

then A is not diag then A is diag

then is not diag

then A is diag

so find eigenvector(s) for the repeated eigenvalue

for $\lambda_1 = 1$, solve $(A - \lambda I)\vec{x} = 0$

$$\left[\begin{array}{ccc|c} 1-\lambda & 2 & -1 & 0 \\ 0 & 1-\lambda & 3 & 0 \\ 0 & 0 & 2-\lambda & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\xrightarrow{\text{RREF}}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable means one eigenvector

so $\lambda_1 = 1$ has only one eigenvector (alg mult 2) because alg mult 1

$\lambda_2 = 2$

so only 2 vectors in total, but would need 3

No, A is not diagonalizable

note: if the geometric mult is equal to algebraic mult for each eigenvalue, then matrix is diagonalizable - otherwise it's not

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example: diagonalize $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

answer: step 1: find eigenvalues

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 6 = 0$$
$$2 - 3\lambda + \lambda^2 - 6 = 0$$
$$\lambda^2 - 3\lambda - 4 = 0$$
$$(\lambda - 4)(\lambda + 1) = 0$$
$$\lambda = -1, 4$$

step 2: find eigenvectors with solving $(A - \lambda I)\vec{x} = 0$

for $\lambda_1 = -1$, $\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & 3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 3/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad x + \frac{3}{2}y = 0$

free variable $y = t$

$$\begin{cases} x = -\frac{3}{2}t \\ y = t \end{cases} \qquad \vec{x}_1 = t\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$$

so eigenvector is $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$

now find second eigenvector:

$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

for $\lambda_2 = 4$ $\quad \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 2 & -2 & 0 \end{array}\right]$

$\xrightarrow{\text{RREF}}$

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad x - y = 0$

let $y = t$

$$\begin{cases} x = t \\ y = t \end{cases} \qquad \vec{x}_2 = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

step 3: write out D and P

so $D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ and $P = \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}$

[note: $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$ also acceptable]

now use your diagonalization of A to calculate A^8

how? $A = PDP^{-1}$

$$A^2 = (PDP^{-1})(PDP^{-1})$$
$$= PD\underbrace{P^{-1}P}_{I}DP^{-1}$$
$$= PD^2P^{-1}$$
$$\downarrow$$
$$A^N = PD^NP^{-1}$$

but $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ so $D^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$= \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$

$$D^N = \begin{bmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{bmatrix}$$

$$P^{-1} = \frac{1}{-5}\begin{bmatrix} 1 & -1 \\ -2 & -3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$$

so if $A = PDP^{-1}$

$$= \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\frac{1}{5}\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$$

then $A^8 = \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \overset{=1}{(-1)^8} & 0 \\ 0 & 4^8 \end{bmatrix}\frac{1}{5}\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$

$$= \frac{1}{5}\begin{bmatrix} -3 & 4^8 \\ 2 & 4^8 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 26215 & 39321 \\ 26214 & 39322 \end{bmatrix}$$

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example: consider the matrix A where

$$\begin{cases} A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\ A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \end{cases}$$

Find matrices P and D and write A in the form $A = PDP^{-1}$.
Then calculate A.

answer: method #1

DO NOT USE! THIS METHOD IS LONG AND HORRIBLE!

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

so $a + 2b = 3$
$c + 2d = 6$

and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

so $a + b = 5$
$c + d = 5$

then solve the 4×4 system to get A, then find eigenvalues and eigenvectors

method #2: USE THIS ONE!

$$A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix} \quad \text{so} \quad \lambda_1 = 3 \text{ and } \vec{x}_1 = \begin{bmatrix}1\\2\end{bmatrix}$$

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix} \quad \text{so} \quad \lambda_2 = 5 \text{ and } \vec{x}_2 = \begin{bmatrix}1\\1\end{bmatrix}$$

then $P = \begin{bmatrix}1 & 1\\2 & 1\end{bmatrix}$ and $D = \begin{bmatrix}3 & 0\\0 & 5\end{bmatrix}$

while $P^{-1} = \frac{1}{-1}\begin{bmatrix}1 & -1\\-2 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1\\2 & -1\end{bmatrix}$

finally
$$\begin{aligned} A &= PDP^{-1} \\ &= \begin{bmatrix}1 & 1\\2 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\0 & 5\end{bmatrix}\begin{bmatrix}-1 & 1\\2 & -1\end{bmatrix} \\ &= \begin{bmatrix}3 & 5\\6 & 5\end{bmatrix}\begin{bmatrix}-1 & 1\\2 & -1\end{bmatrix} \\ &= \begin{bmatrix}7 & -2\\4 & 1\end{bmatrix} \end{aligned}$$