

Section 5.1: Orthogonality

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definition: A set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_k\}$ in \mathbb{R}^n is an orthogonal set iff

$$\vec{v}_i \cdot \vec{v}_j = 0 \quad \text{for all } i \neq j$$

note: $k \leq n$

examples: $\{\hat{i}, \hat{j}, \hat{k}\}$ is an orthogonal set in \mathbb{R}^3

$\{\hat{i}, \hat{k}\}$ " " " "

example: Is the following set of vectors an orthogonal set?

$$\left\{ \vec{v}_1 = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

answer: $\vec{v}_1 \cdot \vec{v}_2 = -4 + 4 + 0 = 0$

$$\vec{v}_1 \cdot \vec{v}_3 = 8 + 2 - 10 = 0$$

yes

$$\vec{v}_2 \cdot \vec{v}_3 = -2 + 2 + 0 = 0$$

theorem: if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_k\}$ is an orthogonal set of vectors in \mathbb{R}^n , then these vectors are LI
- linearly independent

definition: an orthogonal basis of a subspace is a basis that is an orthogonal set

examples: $\{\hat{i}, \hat{j}, \hat{k}\}$ is an orthogonal basis of \mathbb{R}^3

$$\left\{ \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

is an orthogonal basis of \mathbb{R}^3

$$\left\{ \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \begin{bmatrix} -8 \\ 8 \end{bmatrix} \right\}$$

is an orthogonal basis of \mathbb{R}^2

Theorem: Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_k\}$ be an orthogonal basis of subspace ω . For any vector \vec{v} in ω , we have

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + \dots + c_k \vec{v}_k$$

$$\text{with } c_i = \frac{\vec{v} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

Discussion: will not be tested

why? proof:

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + \dots + c_k \vec{v}_k$$

$$\vec{v} \cdot \vec{v}_i = c_1 \vec{v}_1 \cdot \vec{v}_i + c_2 \vec{v}_2 \cdot \vec{v}_i + c_3 \vec{v}_3 \cdot \vec{v}_i + \dots + c_k \vec{v}_k \cdot \vec{v}_i$$

recall $\vec{v}_j \cdot \vec{v}_i = 0$ for $j \neq i$

orthogonal - any two vectors in basis are orthogonal

$$\vec{v} \cdot \vec{v}_i = c_i \vec{v}_i \cdot \vec{v}_i$$

the only term on the right that survives has matching subscripts

$$c_i = \frac{\vec{v} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

example: consider $B = \left\{ \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$,

which is an orthogonal basis of \mathbb{R}^3 .

Find the components of $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$ in basis B .

$$\text{answer: } \vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

$$c_1 = \frac{\vec{v} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \frac{8+2-35}{16+4+25} = \frac{-25}{45} = -\frac{5}{9}$$

$$c_2 = \frac{\vec{v} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} = 0$$

$$c_3 = \frac{\vec{v} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} = \frac{19}{9}$$

$$\vec{v} = -\frac{5}{9} \vec{v}_1 + \frac{19}{9} \vec{v}_3 \quad \Leftarrow \quad [\vec{v}]_B = \begin{bmatrix} -5/9 \\ 0 \\ 19/9 \end{bmatrix}_B$$

definition: a set of vectors in \mathbb{R}^N is called orthonormal iff it is an orthogonal set of unit vectors

so if $S = \{\vec{q}_1, \vec{q}_2, \vec{q}_3, \dots, \vec{q}_k\}$ is orthonormal,

$$\text{then } \vec{q}_i \cdot \vec{q}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

given an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_k\}$,

we can get an orthonormal set by scaling each vector (dividing by its norm/magnitude/length)

example: $\left\{ \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$ is an orthogonal set

Find the corresponding orthonormal set.

answer: divide each vector by its norm

$$\|\vec{v}_1\| = \sqrt{16 + 4 + 25} = \sqrt{45} = 3\sqrt{5}$$

$$\|\vec{v}_2\| = \sqrt{1+4} = \sqrt{5}$$

$$\|\vec{v}_3\| = \sqrt{4+1+9} = 3$$

orthonormal set is $\left\{ \frac{1}{3\sqrt{5}} \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$

Section 5.1: cont'd 2023/11/22

Why are orthonormal sets useful?
transformations of coordinate systems

If $\{\hat{q}_1, \hat{q}_2, \hat{q}_3, \dots, \hat{q}_k\}$ is an orthonormal basis of W , then for all \vec{v} in W we can write

$$\vec{v} = c_1 \hat{q}_1 + c_2 \hat{q}_2 + c_3 \hat{q}_3 + \dots + c_k \hat{q}_k$$

$$\text{where } c_i = \vec{v} \cdot \hat{q}_i \quad \text{for } i=1,2,3,\dots,k$$

why? $\hat{q}_i \cdot \hat{q}_i = 1$, since vectors \hat{q}_i are all unit vectors

orthogonal matrices:

definition: an orthogonal matrix is one whose columns form an orthonormal set

orthogonal matrix:

note: does not have to be square

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

↑ ↑ ↑

each column is orthogonal to every other column



each column is a unit vector

cool property: iff Q is an orthogonal matrix

then

$$Q^{-1} = Q^T$$

digression: will not be tested

proof: for the matrix $Q = [\hat{q}_1 | \hat{q}_2 | \hat{q}_3 | \dots | \hat{q}_k]$

$$Q^T Q = \left[\begin{array}{c} \hat{q}_1 \\ \hat{q}_2 \\ \hat{q}_3 \\ \vdots \\ \hat{q}_k \end{array} \right] [\hat{q}_1 | \hat{q}_2 | \hat{q}_3 | \dots | \hat{q}_k]$$

$$= \begin{bmatrix} \hat{q}_1 \cdot \hat{q}_1 & \hat{q}_1 \cdot \hat{q}_2 & \hat{q}_1 \cdot \hat{q}_3 & \dots & \hat{q}_1 \cdot \hat{q}_k \\ \hat{q}_2 \cdot \hat{q}_1 & \hat{q}_2 \cdot \hat{q}_2 & \dots & & \hat{q}_2 \cdot \hat{q}_k \\ \vdots & \vdots & \ddots & & \vdots \\ \hat{q}_k \cdot \hat{q}_1 & & & \hat{q}_k \cdot \hat{q}_k \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q^T Q = I$$

\uparrow
this must also be Q'

example: Is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ an orthogonal matrix?

answer: note: there are 3 different ways to prove this

here we'll use that if A is orthogonal, then

$$A^T = A^{-1} \text{ and } A^T A = I$$

(we'll do a full solution with all three methods for a different example in a bit)

$$A^T A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

yes

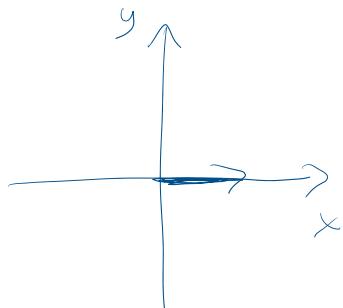
example: is $R(\theta)$ in \mathbb{R}^2 an orthogonal matrix?

answer:

$$\begin{aligned}
 R^T(\theta) R(\theta) &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
 &= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
 \end{aligned}$$

yes

example: consider the transformation in \mathbb{R}^2 which reflects across the y-axis. Is the associated matrix A orthogonal?



$$A\hat{i} = -\hat{i} \quad A\hat{j} = \hat{j}$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

method #1: $A^T A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = I$

yes

method #2: $\vec{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{v}_1 \cdot \vec{v}_2 = 0 \quad \text{so} \quad \vec{v}_1 \perp \vec{v}_2$$

$$\left. \begin{array}{l} \vec{v}_1 \cdot \vec{v}_1 = 1 \\ \vec{v}_2 \cdot \vec{v}_2 = 1 \end{array} \right\} \text{both unit vectors}$$

yes

method #3:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = A^T$$

[yes]

property: the eigenvalues of an orthogonal matrix Q satisfy

$$|\lambda| = 1$$

(this is because orthogonal matrices are "length-preserving" transformations - they don't change the length of the vector they are transforming)

example: show that the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ satisfies $|\lambda| = 1$

answer: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\lambda = \pm \sqrt{-1} = \pm i$$

recall: the absolute value of a complex number \rightarrow the $\sqrt{a^2 + b^2}$.

recall: the absolute value of a complex number is the distance from the origin

