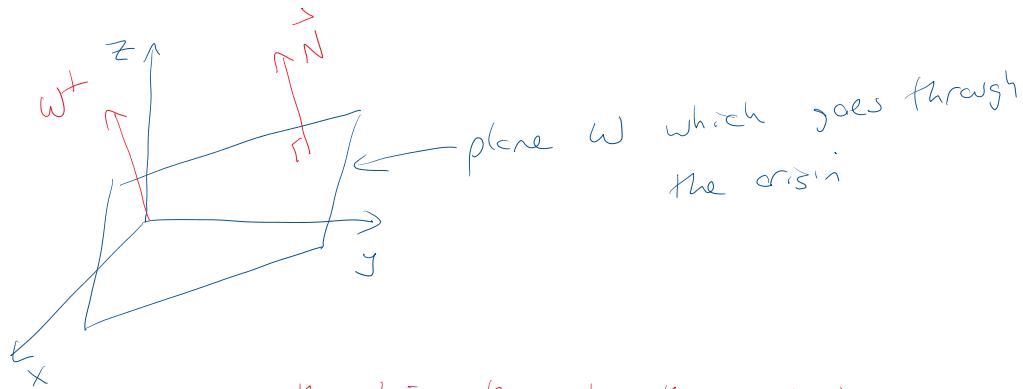


## Section 5.2: Orthogonal Complements and Projections

Wednesday, November 22, 2023 10:15 AM

let us consider a plane  $\omega$  which is a subspace of  $\mathbb{R}^3$  and vector  $\vec{N}$  which is orthogonal to plane  $\omega$



the line through the origin parallel to  $\vec{N}$  is also a subspace and we write it as  $\omega^\perp$

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definition: let  $\omega$  be a subspace. A vector  $\vec{v}$  is orthogonal to  $\omega$  if  $\vec{v}$  is orthogonal to every vector in  $\omega$ .

the set of all vectors orthogonal to  $\omega$  is called the orthogonal complement to  $\omega$  and is denoted by

$\omega^\perp$  (" $\omega$  perp")

if you like,  $\omega^\perp = \{\vec{v} \mid \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \text{ in } \omega\}$

example:  $\omega = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid \begin{array}{l} \text{such that} \\ 3x + 5y + 2z = 0 \end{array} \right\}$

the subspace  $\omega$  find  $\omega^\perp$  the set of all vectors  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  (30) coordinates satisfying this equation

answer:  $w$  is a plane through the origin in  $\mathbb{R}^3$

now the normal  $N = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$  is a vector  $\perp$  to  $w$

$$\text{so } w^\perp = \text{span} \left( \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \right)$$

$$\text{note: } (w^\perp)^\perp = w$$

recall: consider matrix  $A$

we previously learned about the  
row space  $\text{Row}(A)$   
column space  $\text{Col}(A)$

but what is the space of vectors  
orthogonal to  $\text{Row}(A)$ ?  
to  $\text{Col}(A)$ ?

notation  $[\text{Row}(A)]^\perp$  ← what is the set of  
vectors which are all  
 $\perp$  to rows of  $A$

example:  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$  and  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

if  $\vec{x}$  is  $\perp$  to all rows of  $A$ , then

Row 1:  $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$  and same for  
Rows 2 and 3

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

which means that

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



what is this set of vectors  
that you get by solving  
 $A\vec{x} = 0$ ?

null space!  $\text{Null}(A)$

Conclusion:  $[\text{Row}(A)]^\perp = \text{Null}(A)$

but what about the column space,  $\text{Col}(A)$ ?

replace  $A$  by  $A^T$

then  $\text{Row}(A^T) = \text{Col}(A)$

and  $[\text{Col}(A)]^\perp = \text{Null}(A^T)$

summary:  $\text{Row}(A) \perp \text{Null}(A)$       } 4 fundamental spaces of an  $m \times n$  matrix

$\text{Col}(A) \perp \text{Null}(A^T)$

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example: let  $W = \text{span}(\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4)$

where  $\vec{w}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ ,  $\vec{w}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ -3 \\ 1 \end{bmatrix}$ ,  $\vec{w}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ ,  $\vec{w}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Find  $w^\perp$

answer: The vectors are column vectors

If we make matrix  $A$  whose columns are these vectors, then  $w = \text{Col}(A)$

then  $w^\perp = [\text{Col}(A)]^\perp = \text{Null}(A^T)$

$$A^T = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let  $x_2 = s$   
Let  $x_5 = t$   
 $x_1 + x_2 + x_5 = 0$   
 $x_3 + x_5 = 0$   
 $x_4 = 0$

how do we find  $\text{Null}(A^T)$ ? solve  $A^T \vec{x} = \vec{0}$

$$\begin{cases} x_1 = -s - t \\ x_2 = s \\ x_3 = -t \\ x_4 = 0 \\ x_5 = t \end{cases} \quad \vec{x} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

so  $w^\perp = \text{Null}(A^T)$

$$= \text{Span} \left( \left( \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right) \right)$$

note: because there are only 3 leading variables,  $\{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\}$  doesn't form a basis  
(because it has an unnecessary vector)

(because it has an unnecessary vector)

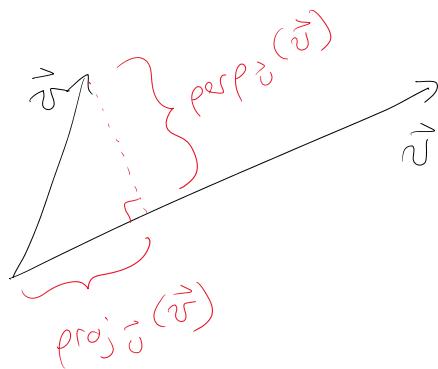
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orthogonal projections:

recall:

projecting  
 $\vec{v}$  onto  $\vec{u}$

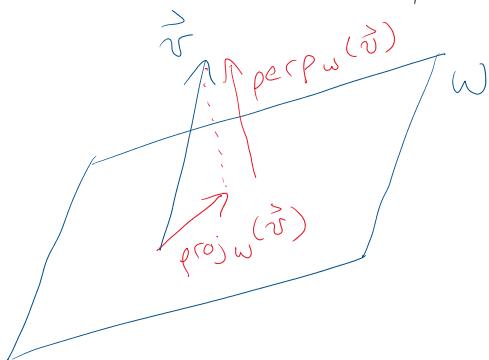


$$\vec{v} = \text{proj}_{\vec{u}}(\vec{v}) + \text{perp}_{\vec{u}}(\vec{v})$$

$\uparrow$   
component of  $\vec{v}$   
parallel to  $\vec{u}$

$\uparrow$   
component of  $\vec{v}$   
perpendicular to  $\vec{u}$

now let's project onto a subspace:



lets say that  $w$   
is a plane

and vector  $\vec{v}$   
is not in that  
plane

note: this is drawn in  $\mathbb{R}^3$ , but applies in  $\mathbb{R}^n$

then

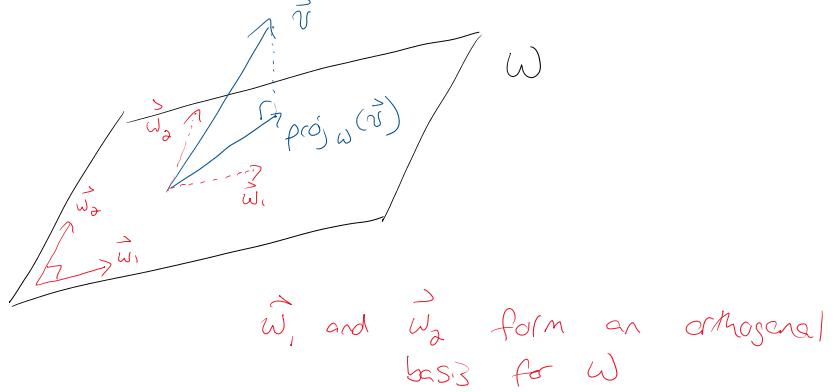
$$\vec{v} = \text{proj}_w(\vec{v}) + \text{perp}_w(\vec{v})$$

any vector  $\vec{v}$  in  $\mathbb{R}^n$  can be expressed uniquely  
in this way

in this way

so, how do we calculate this?

in  $\mathbb{R}^3$ :



$$\text{then } \text{proj}_w(\vec{v}) = \text{proj}_{\vec{w}_1}(\vec{v}) + \text{proj}_{\vec{w}_2}(\vec{v})$$

↑  
Subspace

↑  
first  
vector  
& bcs.

P  
second  
vector  
of bcs.

but ONLY if  $\vec{w}_1 \perp \vec{w}_2$

now how do you get  $\text{perp}_w(\vec{v})$ ?

$$\text{perp}_w(\vec{v}) = \vec{v} - \text{proj}_w(\vec{v})$$

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so the previous example was in  $\mathbb{R}^3$

in general for  $\mathbb{R}^n$

$$\left\{ \begin{array}{l} \text{proj}_w(\vec{v}) = \text{proj}_{\vec{w}_1}(\vec{v}) + \text{proj}_{\vec{w}_2}(\vec{v}) + \dots + \text{proj}_{\vec{w}_k}(\vec{v}) \\ \text{perp}_w(\vec{v}) = \vec{v} - \text{proj}_w(\vec{v}) \end{array} \right.$$

iff  $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_k\}$  is an orthogonal bcs.

example: let  $\omega = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$

Find  $\text{proj } (\vec{v})$  for  $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ .

answer: let  $\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$  and  $\vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

observe that  $\vec{w}_1 \cdot \vec{w}_2 = 0$  so  $\vec{w}_1 \perp \vec{w}_2$

$$\begin{aligned}
 \text{then } \text{proj}_{\omega}(\vec{v}) &= \text{proj}_{\vec{w}_1}(\vec{v}) + \text{proj}_{\vec{w}_2}(\vec{v}) \\
 &= \frac{\vec{w}_1 \cdot \vec{v}}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 + \frac{\vec{w}_2 \cdot \vec{v}}{\vec{w}_2 \cdot \vec{w}_2} \vec{w}_2 \\
 &= \frac{2+1-5}{(-1)+1} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{0+1+5}{0+1+1} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
 &= -\frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} -2/3 \\ 7/3 \\ 11/3 \end{bmatrix}
 \end{aligned}$$

this is the component  
 of  $\vec{v}$  onto  
 subspace  $\omega$

DO NOT SCALE!

Find  $\text{perf}_{\omega}(\vec{v})$ .

$$\begin{aligned}
 \text{perf}_{\omega}(\vec{v}) &= \vec{v} - \text{proj}_{\omega}(\vec{v}) \\
 &= \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} -2/3 \\ 7/3 \\ 11/3 \end{bmatrix}
 \end{aligned}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 8/3 \\ -4/3 \\ 4/3 \end{bmatrix}$$

optional quick check:

$$\text{perp}_\omega(\vec{v}) \cdot \vec{\omega}_1 = \begin{bmatrix} 8/3 \\ -4/3 \\ 4/3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0$$

$$\text{perp}_\omega(\vec{v}) \cdot \vec{\omega}_2 = \begin{bmatrix} 8/3 \\ -4/3 \\ 4/3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$$

note: why can't we scale?

we are trying to find the components of  
 $\vec{v}$  along various vectors

