

Exploration : Cross Product

Monday, September 11, 2023 4:15 PM

$$\text{in } \mathbb{R}^3, \quad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

then the cross product of these vectors

$$\vec{u} \times \vec{v} = \begin{bmatrix} (u_2 v_3 - u_3 v_2) \\ (u_3 v_1 - u_1 v_3) \\ (u_1 v_2 - u_2 v_1) \end{bmatrix} \quad \text{don't memorize!}$$

$\underbrace{\quad}_{\text{single column}}$
which is
a vector

Section : Cross Product

2023/09/12

the shortcut: recall unit vectors $\hat{i} \hat{j} \hat{k}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad \begin{array}{l} \text{we are calculating} \\ \text{the determinant} \\ \text{of this matrix} \\ \text{(more in Chapter 4)} \end{array}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- multiply through the lines ($i u_2 v_3$, for example)

add the $\boxed{\quad}$ $\boxed{\quad}$ $\boxed{\quad}$

add the //
subtract the //

example: compute $\vec{U} \times \vec{V}$ for $\vec{U} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{V} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

answer: $\vec{U} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & 3 & 1 \end{vmatrix}$

$$= \hat{i}(2)(1) - \hat{j}(-1)(2) + \hat{k}(1)(3) - \hat{k}(2)(2) - \hat{i}(-1)(3) - \hat{j}(1)(1)$$

$$= 2\hat{i} - 2\hat{j} + 3\hat{k} - 4\hat{k} + 3\hat{i} - \hat{j}$$

$$\begin{aligned} &= 5\hat{i} - 3\hat{j} - 1\hat{k} \\ &= \begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix} \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{either}$$

note: the result of the cross product is also a vector, one that is perpendicular to the plane containing the first two

("perpendicular" is generally used for two and three dimensions. "orthogonal" is the more general term used for any number of dimensions)

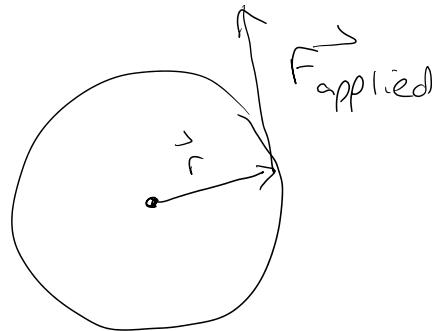
the direction of $\vec{U} \times \vec{V}$ can be found using the right-hand rule

- right hand - place flat with thumb at right-angle

- stick the thumb along the first vector
- fingers along second vector
- then palm pushes in direction of cross product
 $\xrightarrow{\text{fist}} \times \xrightarrow{\text{second}}$

where do we use cross product?

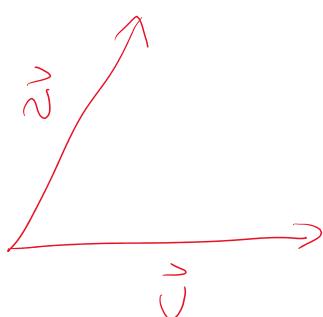
$$\vec{r} = \vec{r} \times \vec{F}$$



flywheel

properties of the cross product:

$$\textcircled{1} \quad \vec{v} \times \vec{w} = - \vec{w} \times \vec{v} \quad (\text{order matters})$$



$\vec{v} \times \vec{w}$ is out of the page by the right-hand rule

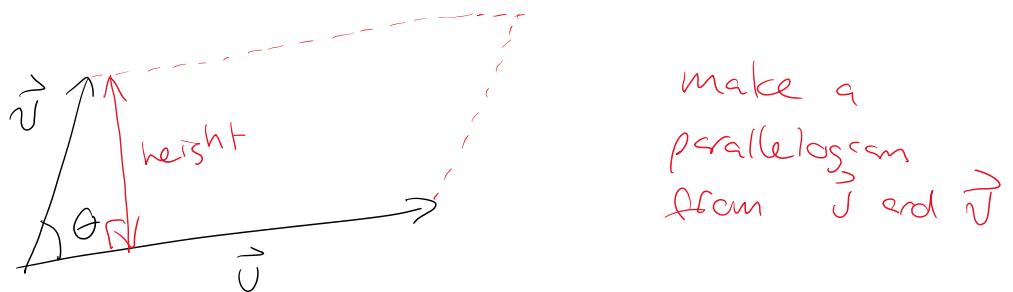
$\vec{v} \times \vec{v}$ is into the page

$$\textcircled{2} \quad \| \vec{v} \times \vec{w} \| = \| \vec{v} \| \| \vec{w} \| \sin \theta$$

where θ is the angle between \vec{u} and \vec{v}

$$[\text{recall: } \vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta]$$

there's a nice physical representation:



If we take \vec{u} as the base of the parallelogram,

$$\text{then height} = \|\vec{v}\| \sin \theta$$

$$\text{so area of parallelogram} = \|\vec{u}\| \|\vec{v}\| \sin \theta$$

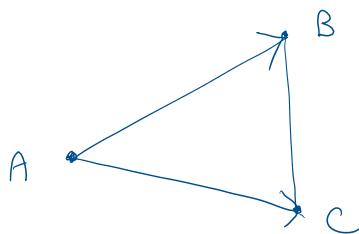
$$= \|\vec{u} \times \vec{v}\|$$

Magnitude of
cross product

(right-hand rule gives you the direction)

example: what is the area of triangle ABC with vertices $A = (1, 0, 1)$, $B = (0, 2, 3)$, and $C = (2, 1, 0)$?

answer:



$$\vec{AB} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{AC} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned}\vec{AB} \times \vec{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 2 \\ -1 & 2 & -1 \end{vmatrix} \\ &= -2\hat{i} + 2\hat{j} - \hat{k} - 2\hat{i} - 2\hat{j} - \hat{k} \\ &= -4\hat{i} + \hat{j} - 3\hat{k} \\ &= \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}\end{aligned}$$

} either is fine

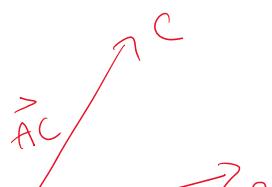
$$\text{area of } \triangle ABC = \frac{1}{2} \|\vec{AB} \times \vec{AC}\|$$

$$\begin{aligned}&= \frac{1}{2} \sqrt{(-4)^2 + (1)^2 + (-3)^2} \\&= \frac{1}{2} \sqrt{26}\end{aligned}$$

example: Find a vector $\vec{N} \perp$ to the plane containing the points

$$A = (-9, 0, 2), \quad B = (1, -3, 1) \quad \text{and} \quad C = (2, -2, 6)$$

answer:



$$\vec{AB} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} \quad \text{and} \quad \vec{AC} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$\vec{AB} = \begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{AC} = \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}$$

↑
vector \vec{N} is
 \perp to the plane
containing \vec{AB} and \vec{AC}

Section: Cross product cont'd 2023/09/13

$$\begin{aligned}\vec{N} &= \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 5 & -1 \\ 6 & -2 & 4 \end{vmatrix} \\ &= -12\hat{i} - 6\hat{j} - 10\hat{k} + 18\hat{k} - 2\hat{i} - 20\hat{j} \\ \vec{N} &= -14\hat{i} - 26\hat{j} + 8\hat{k}\end{aligned}$$

If for some reason we wanted a unit vector,
then:

$$\|\vec{N}\| = \sqrt{(-14)^2 + (-26)^2 + 8^2}$$

$$\hat{N} = \frac{1}{\|\vec{N}\|} \begin{bmatrix} -14 \\ -26 \\ 8 \end{bmatrix} = \frac{1}{\sqrt{936}} \begin{bmatrix} -14 \\ -26 \\ 8 \end{bmatrix}$$

example: Find a unit vector parallel to the yz -plane
that is perpendicular to $\vec{\omega}$, where

$$\vec{\omega} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

method(1): if \vec{v} is the vector that we want,
then $\vec{v} \cdot \vec{\omega} = 0$

and if \vec{v} is parallel to the yz -plane,
then $\vec{v} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$ \in no x -component

$$\vec{v} \cdot \vec{\omega} = 0$$

$$\begin{bmatrix} 0 \\ a \\ b \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 0$$

$$0 + a(-1) + b(2) = 0$$

$$-a + 2b = 0$$

$$a = 2b$$

but \vec{v} is a unit vector

$$\sqrt{0^2 + a^2 + b^2} = 1$$

$$\sqrt{4b^2 + b^2} = 1$$

$$5b^2 = 1$$

$$b^2 = \frac{1}{5}$$

$$b = \pm \frac{1}{\sqrt{5}}$$

be careful! do not say if $b = \pm \frac{1}{\sqrt{5}}$, then $a = \pm \frac{1}{\sqrt{5}}$

b can be
 $+$ or $-$

a can
 $+$ or $-$

but what we want to say is

$$\text{if } b = +\frac{1}{\sqrt{5}} \text{ then } a = +\frac{2}{\sqrt{5}}$$

$$b = -\frac{1}{\sqrt{5}} \quad a = -\frac{2}{\sqrt{5}}$$

so instead, choose $b = \frac{1}{\sqrt{5}}$, then $a = \frac{2}{\sqrt{5}}$

$$\vec{v} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

if you want all unit vectors \perp to $\vec{\omega}$

then write $\vec{v} = \pm \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

method #2: let the vector of interest be \vec{v}

since \vec{v} is \parallel to the yz -plane, then

$\vec{v} \perp$ to x -axis (\hat{i})

and $\vec{v} \perp$ to $\vec{\omega}$

so $\vec{v} \parallel \vec{\omega} \times \vec{e}$

$$\vec{\omega} \times \vec{e} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 2\hat{j} + \hat{k}$$

$$\text{but } \|\vec{\omega} \times \vec{e}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$\text{but } \|\vec{\omega} \times \vec{e}\| = \sqrt{\omega_x^2 + \omega_y^2}$$

$$= \sqrt{5}$$

$$\vec{v} = \pm \frac{\vec{\omega} \times \hat{e}}{\|\vec{\omega} \times \hat{e}\|} = \textcircled{\pm} \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$



can omit, since question
just asked for one vector