

## Review of 5.1 to 7.3

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### Chapter 5:

- def:  $\vec{u}$  and  $\vec{v}$  are orthogonal if  $\vec{u} \cdot \vec{v} = 0$
- every orthogonal set of vectors is LI
- if  $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$  is an orthogonal basis of  $\mathbb{W}$   
and  $\vec{v}$  is in  $\mathbb{W}$ , then

$$[\vec{v}]_B = \begin{bmatrix} \frac{\vec{v} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \\ \frac{\vec{v} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \\ \vdots \\ \text{etc} \end{bmatrix}$$

\* recall: if  $B$  is not orthogonal, find  $[\vec{v}]_B$   
just by solving  
from previous chapter (2?)

$$\left[ \vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3 \mid \dots \mid \vec{v} \right] \text{ and RREF}$$

def:  $\vec{u}$  and  $\vec{v}$  are orthonormal if they  
are orthogonal unit vectors

for  $\vec{u}$ ,  $\frac{\vec{u}}{\|\vec{u}\|}$  is its unit vector

def: an orthogonal matrix has orthonormal  
columns

properties: if  $Q$  is orthogonal

$$1) Q^{-1} = Q^T$$

2) all eigenvalues satisfy  $|\lambda| = 1$

3)  ~~$\|Q\vec{x}\| = \|\vec{x}\|$  and  $Q\vec{x} \cdot Q\vec{y} = \vec{x} \cdot \vec{y}$  for all  $\vec{x}, \vec{y}$~~

· def: the orthogonal complement of a subspace  $W$

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$$W^\perp = \{ \vec{v} \mid \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \text{ in } W \}$$

· for any matrix  $A$ :

$$[\text{row}(A)]^\perp = \text{null}(A)$$

$$[\text{col}(A)]^\perp = \text{null}(A^T)$$

)

to find  $\text{null}(A)$ , solve

$$A\vec{x} = \vec{0}$$

$$[A \mid 0] \xrightarrow{\text{REF}}$$

- if  $\{\vec{w}_1, \vec{w}_2, \vec{w}_3, \dots, \vec{w}_k\}$  is an orthogonal basis of  $W$ , then

$$\text{proj}_W(\vec{v}) = \text{proj}_{\vec{w}_1}(\vec{v}) + \text{proj}_{\vec{w}_2}(\vec{v}) + \dots + \text{proj}_{\vec{w}_k}(\vec{v}) \in W$$

$$\text{perp}_W(\vec{v}) = \vec{v} - \text{proj}_W(\vec{v}) \in W^\perp$$

(I will give you the projection vector)

### Section 5.3

- Gram Schmidt - turn any basis into orthogonal basis so that projection onto subspace can be defined

start with  $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \dots, \vec{x}_k\}$

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2)$$

⋮  
↓

$$\vec{v}_k = \vec{x}_k - \text{proj}_{\vec{v}_1}(\vec{x}_k) - \text{proj}_{\vec{v}_2}(\vec{x}_k) - \dots - \text{proj}_{\vec{v}_{k-1}}(\vec{x}_k)$$

$A = QR$  where  $Q$  is an orthogonal matrix  
(orthonormal columns)  
 $R$  is an invertible upper triangular matrix

we can get a QR factorization for  $A$   
if  $A$  has LI columns

→ to find  $Q$ : apply Gram-Schmidt to the columns of  $A$ , then scale to get orthonormal vectors

→ these are the columns of  $Q$

$$\rightarrow R = Q^T A$$

Section 5.4:

for symmetric matrices ( $A^T = A$ ), we can orthogonally diagonalize

to get  $A = Q D Q^T$  where

$Q$  = orthogonal matrix  
(orthonormal columns)  
 $D$  = diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

where the  $\lambda_i$  are the eigenvalues of  $A$

$$\begin{bmatrix} \vdots & \lambda_n \end{bmatrix}$$

$$Q = \left[ \vec{q}_1 \mid \vec{q}_2 \mid \dots \mid \vec{q}_n \right] \quad \text{where } \vec{q}_i \text{ is an eigenvector for } \lambda_i$$

and  $\{\vec{q}_1, \vec{q}_2, \dots, \vec{q}_n\}$  is orthonormal

- eigenvectors for distinct eigenvalues are perpendicular

- for repeated eigenvalues (alg mult > 1)

- we use Gram Schmidt to orthogonalize the eigenvectors for that eigenvalue

- scale to get unit vectors

### Section 7.3:

For an **inconsistent** system  $A\vec{x} = \vec{b}$   
(no solution)

$$A\vec{x}_{LS} = \text{proj}_{\text{Col}(A)} \vec{b}$$

- $\vec{x}_{LS} = (A^T A)^{-1} A^T \vec{b}$  ← will be given

$$\|\vec{b} - A\vec{x}_{LS}\| = \text{least squares error}$$

- to find the least-squares linear fit for  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$

line:  $y = mx + b$

plug in points:

$$\left\{ \begin{array}{l} y_1 = mx_1 + b \\ y_2 = mx_2 + b \\ \vdots \\ y_n = mx_n + b \end{array} \right.$$

System of equations in  $m$  and  $b$

$$\left\{ \begin{array}{l} y_1 = mx_1 + b \\ y_2 = mx_2 + b \\ \vdots \\ y_n = mx_n + b \end{array} \right.$$

System of  
equations  
in m and b

this yields a matrix equation

$$\left[ \begin{array}{c|c} x_1 & | \\ x_2 & | \\ \vdots & | \\ x_n & | \end{array} \right] \left[ \begin{array}{c} m \\ b \end{array} \right] = \left[ \begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array} \right]$$

$A \quad \vec{x} \quad \vec{b}$

-solve for  $\vec{x}_{LS}$  to read off m and b