

Review for Final Exam:

Wednesday, December 06, 2023 9:45 AM

Consider the two points $P = (2, 0, 1)$ and $Q = (1, 2, 2)$.

- Calculate the length of vector \vec{PQ} .
- Is \vec{PQ} a unit vector? Explain briefly.

answer: a) $\vec{PQ} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$$\|\vec{PQ}\| = \sqrt{(-1)^2 + 2^2 + 1^2} = \sqrt{6}$$

b) No, $\sqrt{6} \neq 1$ or magnitude $\neq 1$.

example: Consider the following three points:

$$A = (0, 1, -2), \quad B = (1, 2, 1), \quad C = (-1, 2, -3)$$

- calculate the angle $0^\circ \leq \theta \leq 180^\circ$ between vectors \vec{AB} and \vec{AC} .
- calculate the vector component of \vec{AC} along \vec{BC} .

answer: a) $\vec{AB} \cdot \vec{AC} = \|\vec{AB}\| \|\vec{AC}\| \cos \theta$

$$\cos \theta = \frac{\vec{AB} \cdot \vec{AC}}{\|\vec{AB}\| \|\vec{AC}\|}$$

$$\vec{AB} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad \vec{AC} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\cos \theta = \frac{-3}{\sqrt{11} \sqrt{3}} = 121.5^\circ$$



$$\text{proj}_{\vec{BC}}(\vec{AC}) = \vec{BC} \cdot \vec{AC} \frac{\vec{AC}}{\|\vec{AC}\|}$$

b)

$$\text{proj}_{\vec{BC}}(\vec{AC}) = \frac{\vec{BC} \cdot \vec{AC}}{\vec{BC} \cdot \vec{BC}} \vec{BC}$$

$$= \frac{2+4}{4+16} \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix}$$

$$= \frac{3}{10} \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix} = \frac{-3}{5} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\vec{AC} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{BC} = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix}$$

example: Give an equation in general form for the plane passing through the point $P = (3, -2, 1)$ and perpendicular to the line with parametric equations

$$\begin{cases} x = 4 - 2t \\ y = -2 + t \\ z = 3 - 5t \end{cases}$$

answer: so this line goes through the point $(4, -2, 3)$ and has direction vector

$$\begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$$

then the normal to the plane is parallel to

$$\begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$$

$$ax + by + cz = d$$

$$-2x + y - 5z = d$$

now plug in $(3, -2, 1)$:

$$-2(3) + (-2) - 5(1) = d$$

$$d = -13$$

$$-2x + y - 5z = -13$$

Review: cont'd 2023/12/08

Example: Consider vector \vec{v} and subspace W .

$$\vec{v} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}, \quad W = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}\right)$$

- a) Find an orthogonal basis for W
- b) For the basis you found in part (a), express \vec{v} as a linear combo of the vectors in that basis.

answer: a) Gram-Schmidt: Let the orthogonal basis be $\{\vec{w}_1, \vec{w}_2\}$

then $\vec{w}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$$\begin{aligned} \text{and } \vec{w}_2 &= \vec{x}_2 - \text{proj}_{\vec{w}_1}(\vec{x}_2) = \vec{x}_2 - \frac{\vec{w}_1 \cdot \vec{x}_2}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 \\ &= \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} - \frac{3-1-4}{1+1+4} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 10/3 \\ 2/3 \\ -4/3 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 10 \\ 2 \\ -4 \end{bmatrix} \text{ or } \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$$

so orthogonal basis for W is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} \right\}$

$$b) \vec{v} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} = c_1 \vec{\omega}_1 + c_2 \vec{\omega}_2$$

$$\text{where } c_1 = \frac{\vec{\omega}_1 \cdot \vec{v}}{\vec{\omega}_1 \cdot \vec{\omega}_1} = \frac{-1}{6}$$

$$c_2 = \frac{\vec{\omega}_2 \cdot \vec{v}}{\vec{\omega}_2 \cdot \vec{\omega}_2} = \frac{25}{30} = \frac{5}{6}$$

example: Consider the following subspace:

$$W = \text{span} \left(\begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right)$$

Find a basis for W^\perp

answer: W is a set of columns, so we're looking at the column space of A

$$W^\perp = \text{Null}(A^T)$$

$$A^T = \begin{bmatrix} 1 & 3 & 2 & -2 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 5 & -5 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$\uparrow \quad \uparrow$
let $x_3 = s$ let $x_4 = t$

$$\begin{cases} x_1 = -5s + 5t \\ x_2 = s - t \\ x_3 = s \\ x_4 = t \end{cases}$$

$$\vec{x} = s \begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

so basis for $w^{\perp} = \left\{ \begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Setting up word problems:

- ① find the parabola $y = ax^2 + bx + c$ that goes through the points $(1, -2), (-1, 8), (2, -1)$

setup: plug in points:

$$\begin{array}{lll} (1, -2) \text{ gives } & -2 = a + b + c \\ (-1, 8) \text{ gives } & 8 = a - b + c \\ (2, -1) \text{ gives } & -1 = 4a + 2b + c \end{array}$$

- ② chemical equation



atoms: $\text{LHS} = \text{RHS}$

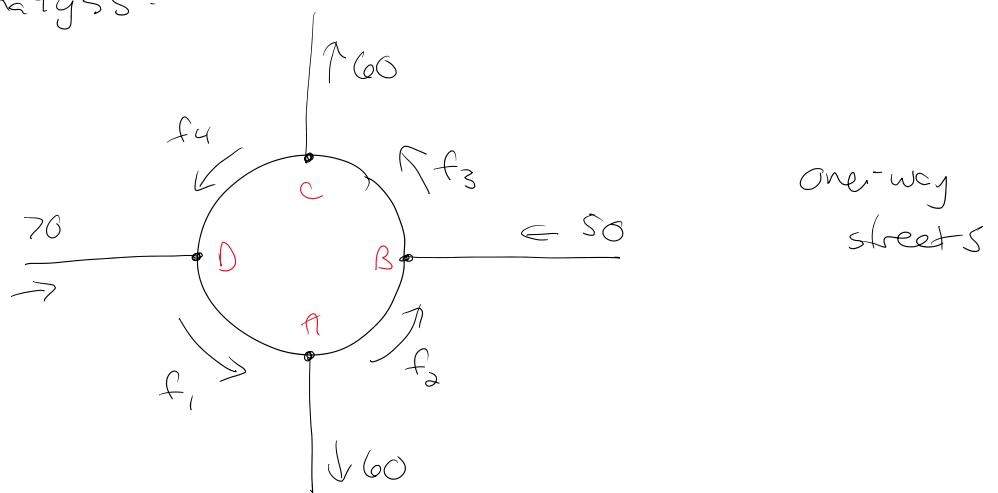
$$\begin{array}{ll} \text{N:} & a = 2c \\ \text{H:} & 3a = 2d \\ \text{O:} & 2b = d \end{array}$$

$$\left\{ \begin{array}{l} a - 2c = 0 \\ 3a - 2d = 0 \\ 2b - d = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} a = \frac{2}{3}t \\ b = \frac{1}{3}t \\ c = \frac{1}{3}t \\ d = t \end{array} \right.$$

\Rightarrow
So choose $t = 6$ and find a, b, c, d

③ network analysis:



setup: at any junction, incoming = outgoing

$$A: f_1 = f_2 + 60$$

$$B: f_2 + 50 = f_3$$

$$C: f_3 = f_4 + 60$$

$$D: f_4 + 70 = f_1$$

④

Twenty coins make exactly \$3 and consist of nickels (5¢), dimes (10¢), and quarters (25¢).

Setup

$$n + d + q = 20$$

$$5n + 10d + 25q = 300$$