Section 4.4: Diagonalization

Tuesday, November 14, 2017 1:56 PM

definition: a square matrix A is diagonalizable if there is a diagonal matrix D and an invertible matrix P such that

$$P^{-1}AP = D$$

or

$$A = PDP^{-1}$$

example: we found earlier that the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

has $\lambda_1 = 2$ with $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$ with $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

then $D = \begin{array}{cc} \lambda_1 & \\ \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \\ & \lambda_2 \end{array}$ and $P = \begin{array}{c} \begin{array}{cc} \vec{x}_1 & \vec{x}_2 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$

note: could also have $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

check: $A = PDP^{-1}$ where $P^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \quad \checkmark$$

theorem: An $N \times N$ matrix A is diagonalizable if A has N linearly independent eigenvectors

example: a 3×3 matrix has two eigenvalues (one is repeated) but does have 3 distinct eigenvectors

$\rightarrow$ diagonalizable

a 3×3 matrix has two eigenvalues and two eigenvectors

$\rightarrow$ not diagonalizable

proof: suppose we have N linearly independent eigenvectors $\vec{x}_1, \vec{x}_2, \vec{x}_3 \ldots \vec{x}_N$ corresponding to eigenvalues $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_N$

$$A\vec{x}_1 = \lambda_1 \vec{x}_1$$
$$A\vec{x}_2 = \lambda_2 \vec{x}_2$$
$$\vdots$$
$$A\vec{x}_N = \lambda_N \vec{x}_N$$

$$AP = PD \quad \Longleftrightarrow \quad A = PDP^{-1}$$

if $P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_N \end{bmatrix}$ the eigenvectors are the columns of P

and $D = \begin{bmatrix} \lambda_1 & & & \text{all zeros} \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots & \\ \text{all zeros} & & & & \lambda_N \end{bmatrix}$

then $AP = A \left[\vec{x}_1 \,\middle|\, \vec{x}_2 \,\middle|\, \vec{x}_3 \,\middle|\, \cdots \,\middle|\, \vec{x}_N \right]$

$$= \left[A\vec{x}_1 \,\middle|\, A\vec{x}_2 \,\middle|\, A\vec{x}_3 \,\middle|\, \cdots \quad A\vec{x}_N \right]$$

$$= \left[\lambda_1 \vec{x}_1 \,\middle|\, \lambda_2 \vec{x}_2 \,\middle|\, \lambda_3 \vec{x}_3 \,\middle|\, \cdots \quad \lambda_N \vec{x}_N \right]$$

$$= \underbrace{\left[\vec{x}_1 \,\middle|\, \vec{x}_2 \,\middle|\, \vec{x}_3 \,\middle|\, \cdots \quad \vec{x}_N \right]}_{P} \underbrace{\begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_N \end{bmatrix}}_{D}$$

example: diagonalize $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

answer:

step 1: find eigenvalues

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 6 = 0$$

$$2 - 3\lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda+1)(\lambda-4) = 0$$

$$\lambda = -1, 4$$

step 2: find eigenvectors: solve $(A - \lambda_1 I) = 0$

$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

for $\lambda_1 = -1$

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & 3 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$y = t$

so $\begin{cases} x = -\frac{3}{2}t \\ y = t \end{cases}$

$$\vec{x}_1 = t\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$$

so $\vec{x} = \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$