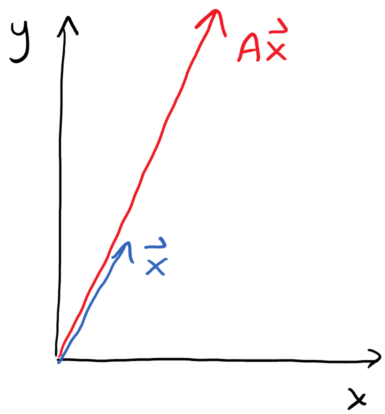
Section 3.6: Intro to Linear Transformations

Wednesday, October 24, 2018 11:41 AM

example: let $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

then $A\vec{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$



A transformed $\vec{x}$
- by rotating and scaling it

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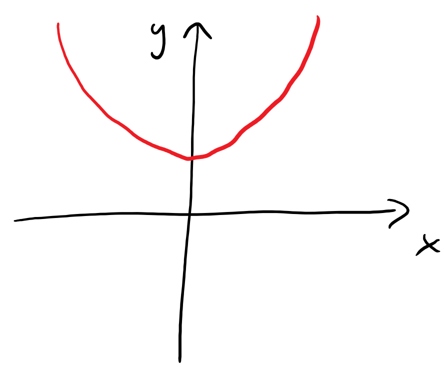
notation: if $f(x) = x^2+1$



<u>input</u>	<u>output</u>
x	y
⇑	⇑
in $\mathbb{R}$	in $\mathbb{R}$

$$f : \mathbb{R} \to \mathbb{R}$$

set of inputs

set of outputs

domain: $\mathbb{R}$ inputs of f

codomain: $\mathbb{R}$ general set of outputs

range: $[1, \infty)$ the set of all outputs of f

or $\{ y \mid y \geq 1 \}$

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $f(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

then $f: \mathbb{R}^2 \to \mathbb{R}^2$

i.e. input and output of f are both in $\mathbb{R}^2$

In general, if A is an $M \times N$ matrix:

$$T(\vec{x}) = \underset{M \times N}{A} \, \underset{N \times 1}{\vec{x}}$$

then $T: \mathbb{R}^N \to \mathbb{R}^M$

$\vec{x}$ is $N \times 1$ $A\vec{x}$ is $M \times 1$

example: let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 0 & -1 \end{bmatrix}$ and define $T_A(\vec{x}) = A\vec{x}$

Find the domain, codomain, and range of T_A.

answer:

$$\underset{3\times 2}{A}\ \underset{\underline{2}\times 1}{\vec{x}}$$

for this product to be defined, $\vec{x}$ must be 2×1

so $\boxed{\text{domain is } \mathbb{R}^2}$

codomain: $\underset{3\times 2}{A}\underset{2\times 1}{\vec{x}}$ has output 3×1

so $\boxed{\text{codomain is } \mathbb{R}^3}$

notation: T_A : domain $\to$ codomain

$T_A : \mathbb{R}^2 \to \mathbb{R}^3$

range: set of actual outputs

$$T_A(\vec{x}) = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+3y \\ x+4y \\ -y \end{bmatrix}$$

$$= \underbrace{x\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}}$$

since x, y can be any scalars, this is Col(A)

so $\boxed{\text{range of } T_A \text{ is } \mathrm{Col}(A)}$

definition: A transformation $T: \mathbb{R}^N \to \mathbb{R}^M$ is <u>linear</u> if

i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^N$

ii) $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u}$ in $\mathbb{R}^N$ and all scalars c

note: If $T: \mathbb{R}^N \to \mathbb{R}^M$ has the form

$T(\vec{x}) = A\vec{x}$ for some matrix A which is $M \times N$,

then T is linear.

why? i) $\underbrace{T(\vec{u} + \vec{v})}_{\text{function notation}} = \overbrace{A(\vec{u} + \vec{v})}^{\text{multiplication with brackets}}$

$= A\vec{u} + A\vec{v}$ from matrix multiplication

$= T(\vec{u}) + T(\vec{v})$ ✓

ii) $T(c\vec{x}) = A(c\vec{x}) = cA\vec{x}$
$= cT(\vec{x})$ ✓

example: Show that $T: \mathbb{R}^2 \to \mathbb{R}^3$ where

$T(x, y) = (4x+5y,\ 2x,\ 3x-6y)$ is linear.

$$T(x,y) = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$= \begin{bmatrix} 4x+5y \\ 2x \\ 3x-6y \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} x + \begin{bmatrix} 5 \\ 0 \\ -6 \end{bmatrix} y$$

$$= \underbrace{\begin{bmatrix} 4 & 5 \\ 2 & 0 \\ 3 & -6 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

so $T(\vec{x}) = A\vec{x}$ where A is a matrix

so $T(\vec{x})$ is linear