Section 5.1: Orthogonality

Monday, November 19, 2018 11:57 AM

definition: A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ in $\mathbb{R}^N$ is an orthogonal set if and only if

$$\vec{v}_i \cdot \vec{v}_j = 0 \quad \text{for all } i \neq j$$

note: $k \leq N$

example: $\{\hat{i}, \hat{j}, \hat{k}\}$ is an orthogonal set of vectors in $\mathbb{R}^3$

note: $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \dots \vec{e}_N\}$ is an orthogonal set in $\mathbb{R}^N$

unit vectors in $\mathbb{R}^N$

example: Is the following set of vectors an orthogonal set?

$$\left\{ \vec{v}_1 = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

answer:

$$\vec{v}_1 \cdot \vec{v}_2 = -4 + 4 = 0$$
$$\vec{v}_1 \cdot \vec{v}_3 = 8 + 2 - 10 = 0$$
$$\vec{v}_2 \cdot \vec{v}_3 = -2 + 2 = 0$$

✓

$\boxed{\text{Yes}}$

theorem: If $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is an orthogonal set of vectors in $\mathbb{R}^N$, then they're LI.

LI = linearly independent

definition: An <u>orthogonal basis</u> of a subspace is a basis that is an orthogonal set.

examples: $\{\hat{i}, \hat{j}, \hat{k}\}$ is an orthogonal basis of $\mathbb{R}^3$

$\left\{ \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$ is also an orthogal basis of $\mathbb{R}^3$

$\left\{ \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ -8 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^2$

theorem: let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots \vec{v}_k\}$ be an orthogonal basis of a subspace W. For any vector $\vec{v}$ in W, we have

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \dots c_k\vec{v}_k$$

$$\text{with} \quad c_i = \frac{\vec{v}\cdot\vec{v}_i}{\vec{v}_i\cdot\vec{v}_i}$$

proof:

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \dots c_k\vec{v}_k$$

$\vec{v} \cdot \vec{v}_i = c_1 \vec{v}_1 \cdot \vec{v}_i + c_2 \vec{v}_2 \cdot \vec{v}_i + \ldots \quad c_k \vec{v}_k \cdot \vec{v}_i$

recall $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$

so $\vec{v} \cdot \vec{v}_i = c_i \vec{v}_i \cdot \vec{v}_i$