Section 5.3: Gram-Schmidt Process

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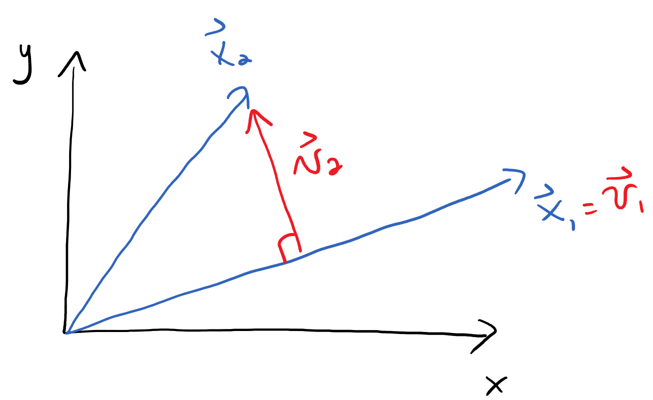
this process is used to find an orthogonal basis of subspace W

begin with a basis $\{\vec{x}_1, \vec{x}_2, \vec{x}_3 \dots \vec{x}_k\}$ of W

and "orthogonalize" it one vector at a time using projections

here's the idea:



$\{\vec{x}_1, \vec{x}_2\}$ is a basis of $\mathbb{R}^2$

we want an orthogonal basis $\{\vec{v}_1, \vec{v}_2\}$

let $\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2)$

note: if you want an orthonormal basis, then divide each vector by its norm

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Gram-Schmidt process:

start from a basis $\{\vec{x}_1, \vec{x}_2, \vec{x}_3 \dots \vec{x}_k\}$ of W

then $\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2)$

$\vec{v}_3 = \vec{x}_3 - \text{proj}_{\vec{v}_1}(\vec{x}_3) - \text{proj}_{\vec{v}_2}(\vec{x}_3)$

$\vdots$

$\vec{v}_k = \vec{x}_k - \text{proj}_{\vec{v}_1}(\vec{x}_k) - \text{proj}_{\vec{v}_2}(\vec{x}_k) \dots - \text{proj}_{\vec{v}_{k-1}}(\vec{x}_k)$

so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3 \dots \vec{v}_k\}$ is an orthogonal basis of W

(if needed, normalize vectors to get an orthonormal basis)

example: Apply the Gram-Schmidt process to transform the following vectors into an orthonormal basis.

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

answer: let $\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

then $\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2)$

$= \vec{x}_2 - \left(\dfrac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \left(\frac{1-2}{1+1} \right) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 \\ 2 \\ 3/2 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \mathrm{proj}_{\vec{v}_1}(\vec{x}_3) - \mathrm{proj}_{\vec{v}_2}(\vec{x}_3)$$

$$= \vec{x}_3 - \frac{\vec{v}_1 \cdot \vec{x}_3}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{x}_3}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \frac{1+1}{1+1} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{3+4-3}{9+16+9} \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{2}{17} \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -6/17 \\ 9/17 \\ -6/17 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$$

so $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} \right\}$ is an <u>orthogonal</u> basis and $\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{34}}\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, \frac{1}{\sqrt{17}}\begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} \right\}$

is the associated <u>orthonormal</u> basis