# Section 5.4: cont'd

Thursday, November 29, 2018 1:32 PM

example: Find an orthogonal matrix $Q$ and diagonal matrix $D$ such that $Q^T A Q = D$ for

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

which has eigenvalues $2, 2, 8$.

eigenvectors: $\lambda_1 = 2$

solve $(A - \lambda I)\vec{x} = \vec{0}$

let $y = s$ let $z = t$

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x = -s - t \\ y = s \\ z = t \end{cases} \quad \text{so} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

so eigenvectors for $\lambda_1 = 2$: $\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

orthogonal? NO! so we'll have to Gram-Schmidt eventually now $\lambda_2 = 8$

solve $(A - \lambda I)\vec{x} = 0$

let $z = t$

$$\left[\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x = t \\ y = t \\ z = t \end{cases} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and $\vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for $\lambda_2 = 8$

now use Gram - Schmidt for $\{\vec{x}_1, \vec{x}_2\}$

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}(\vec{x}_2)$$

$$= \vec{x}_2 - \frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}(\vec{v}_1)$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1}{2}\right) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$= [\ -\tfrac{1}{2}$ ... scale $[\ -1$ ...

$$= \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

so $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\|\vec{v}_1\| = \sqrt{2}$ $\quad \|\vec{v}_2\| = \sqrt{6}$ $\quad \vec{v}_3 = \sqrt{3}$

$$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$