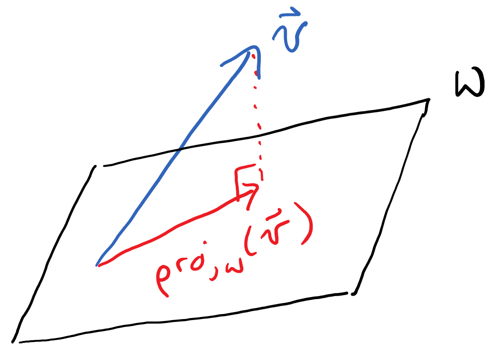
Section 7.3: Least-Squares Approximation

Monday, December 3, 2018 12:01 PM

Best Approximation Theorem:

if W is a subspace of $\mathbb{R}^n$ and $\vec{v}$ is a vector in $\mathbb{R}^n$ (which may or may not be within subspace W) then the best approximation to $\vec{v}$ in W is the projection of $\vec{v}$ onto W

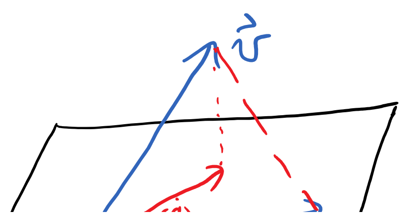


note: $\underbrace{\|\vec{v} - \text{proj}_W(\vec{v})\|}_{\text{Euclidean distance from } \vec{v} \text{ to } W}$ $= \text{perp}_W(\vec{v})$

for any vector $\vec{u}$ in W,

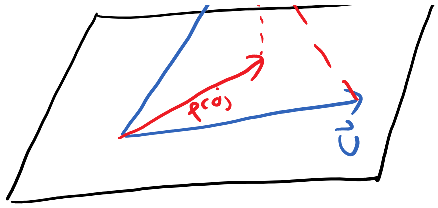
$$\|\vec{v} - \text{proj}_W(\vec{v})\| \leq \|\vec{v} - \vec{u}\|$$

the LHS is the "vertical"



the RHS is the distance, from ... to

the LHS is the "vertical" or shortest distance to the plane



the RHS is the distance from the tip of $\vec{v}$ to any point in the plane

why do we care? we can use this idea to get an approximate solution to inconsistent systems.

example: let $\vec{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$

Find the best approximation to $\vec{v}$ in the plane $W = \text{span}(\vec{w}_1, \vec{w}_2)$ and find the Euclidean distance from $\vec{v}$ to W.

answer: let's first note that $\vec{w}_1 \perp \vec{w}_2$ (otherwise we'd need Gram Schmidt)

$$\text{proj}_W(\vec{v}) = \text{proj}_{\vec{w}_1}(\vec{v}) + \text{proj}_{\vec{w}_2}(\vec{v})$$

$$= \frac{\vec{w}_1 \cdot \vec{v}}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 + \frac{\vec{w}_2 \cdot \vec{v}}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2$$

$$= \frac{10}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \frac{9}{6}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= 2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 11/2 \\ -1 \\ 3/2 \end{bmatrix}$$