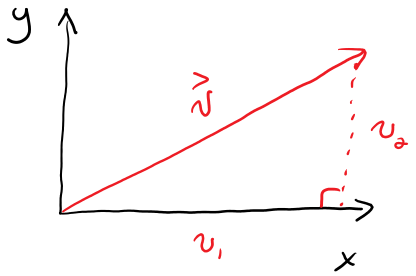
Section 1.2 : Length and Angle : The Dot Product

Thursday, September 08, 2022 9:34 AM

consider the vector $\vec{v}$ in $\mathbb{R}^2$



$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

what is the length of $\vec{v}$?
- called the norm (math) or magnitude (science / tech)

notation : norm of $\vec{v} = \|\vec{v}\|$

note : in science / engineering, usually write $|\vec{v}|$ or v

now

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2} \quad \text{for vectors in } \mathbb{R}^2$$

example : Find the distance between the two points $A = (1, 2, 3)$ and $B = (-1, 1, 2)$.

answer : consider the vector

$$\vec{AB} = \vec{B} - \vec{A}$$

$$= \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$$

$$\|\vec{AB}\| = \sqrt{(-2)^2 + (-1)^2 + (-1)^2}$$

$$= \sqrt{6}$$

the dot product

definition $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$ in $\mathbb{R}^2$

$= u_1 v_1 + u_2 v_2 + \dots u_n v_n$ in $\mathbb{R}^n$

note: $\vec{u} \cdot \vec{v}$ is a scalar (number/constant), <u>not</u> a vector

the dot product is not defined if $\vec{u}$ and $\vec{v}$ have different numbers of components

example: calculate $\vec{u} \cdot \vec{v}$ if $\vec{u} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$

answer: $\vec{u} \cdot \vec{v} = 1(-2) + 5(7)$

$= 33$

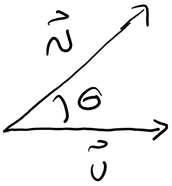
another relationship, derived from the law of cosines:

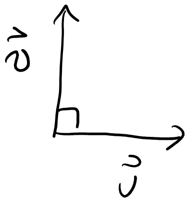
$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \, \cos\theta$$

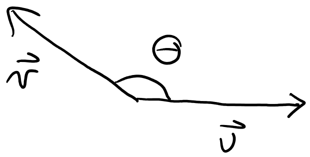
↑

θ is the angle between $\vec{u}$ and $\vec{v}$

properties of the dot product:







$\vec{u}$	$\vec{u}$	$\vec{u}$
θ is acute	θ is right	θ is obtuse
$\vec{u} \cdot \vec{v} > 0$	$\vec{u} \cdot \vec{v} = 0$	$\vec{u} \cdot \vec{v} < 0$

where do we see this in physics? work $W = \vec{F} \cdot \vec{d}$

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example: what is the angle between

$$\vec{u} = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$$

answer: $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

where $\vec{u} \cdot \vec{v} =$

$$3(0) + (-4)(5) + (-1)(2)$$
$$= -22$$

$$\|\vec{u}\| = \sqrt{3^2 + (-4)^2 + (-1)^2}$$
$$= \sqrt{26}$$

$$\|\vec{v}\| = \sqrt{29}$$

$$\cos\theta = \frac{-22}{\sqrt{26}\sqrt{29}}$$

$\theta = 2.5$ rads or 143°

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example: consider the vectors $\vec{u} = \begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

a) Find the value of x if vectors $\vec{u}$ and $\vec{v}$ are perpendicular

b) Find all possible values of x if the angle between $\vec{u}$ and $\vec{v}$ is 60°

answer:

a) $\vec{u} \perp \vec{v}$ means $\vec{u} \cdot \vec{v} = 0$

$$x(1) + 0(2) + 1(2) = 0$$
$$x + 2 = 0$$
$$x = -2$$

b) $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$

$$\vec{u} \cdot \vec{v} = x + 2$$
$$\|\vec{u}\| = \sqrt{x^2+1}$$
$$\|\vec{v}\| = \sqrt{9} = 3$$
$$\cos 60^\circ = \tfrac{1}{2}$$

$$x + 2 = \sqrt{x^2+1} \; 3 \cdot \tfrac{1}{2}$$

$$2(x+2) = 3\sqrt{x^2+1}$$

$$4(x^2+4x+4) = 9(x^2+1)$$

$$5x^2 - 16x - 7 = 0$$

$$x \approx 3.59 \quad \text{or} \quad -0.39$$

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example: show that $\left\{ \begin{array}{l} A = (3,0,2) \\ B = (4,3,0) \\ C = (8,1,-1) \end{array} \right.$ are vertices of a right triangle

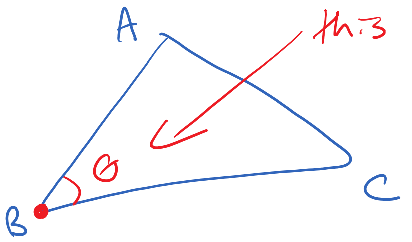
At which point is the right angle?

answer: $\vec{AB} = \begin{bmatrix}1\\3\\-2\end{bmatrix}$, $\vec{AC} = \begin{bmatrix}5\\1\\-3\end{bmatrix}$, $\vec{BC} = \begin{bmatrix}4\\-2\\-1\end{bmatrix}$

$$\vec{AB}\cdot\vec{AC} = 1(5) + 3(1) + (-2)(-3) = 14 \neq 0$$

$$\vec{AB}\cdot\vec{BC} = 1(4) + 3(-2) + (-2)(-1) = 0$$

$$\cancel{\vec{AC}\cdot\vec{BC}}$$

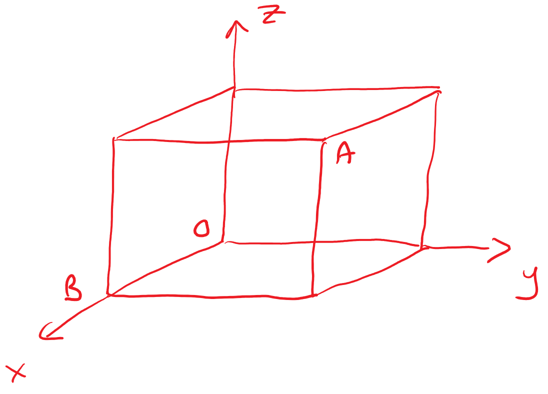


this is actually the right angle

$\vec{AB} \perp \vec{BC}$

the right angle is at point B

example: Find the angle between the interior diagonal of a cube and one of its adjacent sides.



coords of A = (1, 1, 1)
B = (1, 0, 0)

$\vec{OA} = \vec{A} = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\vec{OB} = \vec{B} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$$\vec{A}\cdot\vec{B} = \|\vec{A}\|\,\|\vec{B}\|\cos\theta$$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|\,\|\vec{B}\|}$$

where $\vec{A}\cdot\vec{B} = 1$
$\|\vec{A}\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

$= \frac{1}{\sqrt{3}}$ $\|\vec{B}\| = 1$

$\theta = 54.74^\circ$

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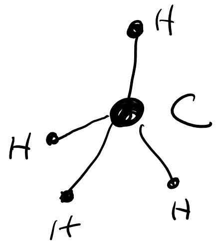
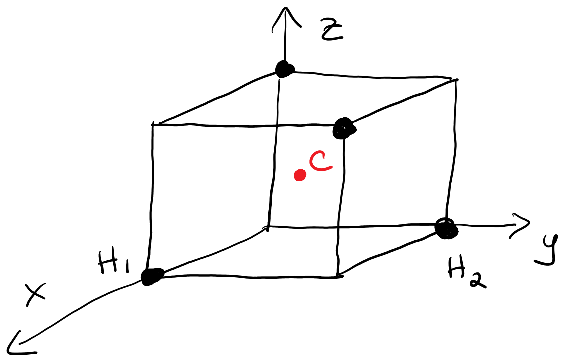
example: CH_4 (methane) has a tetrahedral shape where the hydrogens H are evenly spaced around the central carbon C

what is the angle between the hydrogen bonds?





C is in centre of box

$C = (1, 1, 1)$
$H_1 = (2, 0, 0)$
$H_2 = (0, 2, 0)$

$$\vec{CH_1} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad \vec{CH_2} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\cos\theta = \frac{\vec{CH_1} \cdot \vec{CH_2}}{\|\vec{CH_1}\| \; \|\vec{CH_2}\|} = \frac{-1}{\sqrt{3}\sqrt{3}} = -\frac{1}{3}$$

$\theta = 109.47^\circ$

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properties of the dot product:

① $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

② $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

③ $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$

④ $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

why? $\vec{u} \cdot \vec{u} = u_1 \cdot u_1 + u_2 \cdot u_2 + \dots u_n \cdot u_n$

⑤ Cauchy - Schwartz inequality

$$\underbrace{|\vec{u} \cdot \vec{v}|}_{\text{absolute value bars}} \leq \|\vec{u}\| \, \|\vec{v}\|$$

note: $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$

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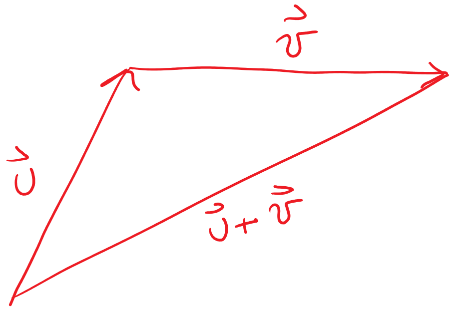
⑥ triangle inequality (not actually a dot product property, but useful nonetheless)



$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

$$\|u+v\| = \|u\| + \|v\|$$

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unit vector: has magnitude (norm) of one

note: in $\mathbb{R}^2$, unit vectors in 'standard position' have their end points on the unit circle

notation: in $\mathbb{R}^2$, unit vectors that lie along the x- and y-axes are called

I will use

x: $\vec{i}$ or $\boxed{\begin{matrix}\hat{i}\\ \hat{j}\end{matrix}}$ or $\vec{e}_1$

y: $\vec{j}$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\vec{e}_2$

"i-hat"

"j-hat"

commonly used in science/engineering

your textbook (ick!)

if in $\mathbb{R}^3$, add $\vec{k}$

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in $\mathbb{R}^2$, $\hat{i} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\hat{j} = \begin{bmatrix}0\\1\end{bmatrix}$

in $\mathbb{R}^3$, $\hat{i} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\hat{j} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\hat{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

so, given a vector $\vec{v} \neq 0$ in $\mathbb{R}^n$, we can find a unit vector that points in the same direction

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

this is called "normalizing" $\vec{v}$

example: normalize $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

answer: $\|\vec{v}\| = \sqrt{(2)^2 + (-1)^2 + (4)^2}$

$= \sqrt{21}$

then unit vector $\hat{v} = \frac{1}{\sqrt{21}} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

or, if you insist, $= \begin{bmatrix} 2/\sqrt{21} \\ -1/\sqrt{21} \\ 4/\sqrt{21} \end{bmatrix}$

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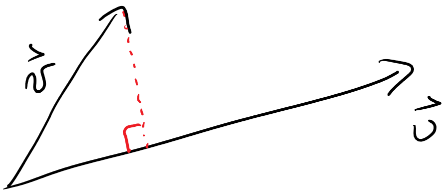
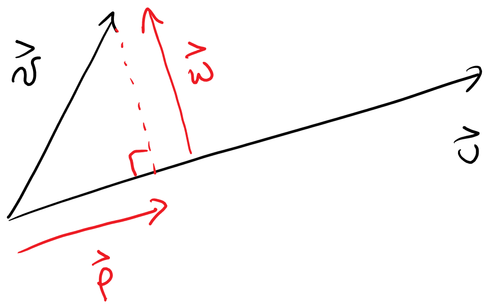
projection operator



consider two vectors $\vec{u}$ and $\vec{v}$. For some reason, we'd like to break $\vec{v}$ into two components

- one along $\vec{u}$ (in same $\overset{\text{parallel to}}{\text{direction}}$ as $\vec{u}$)
- one perpendicular to $\vec{u}$



$\vec{p} = \text{proj}_{\vec{u}}(\vec{v})$ = projection onto $\vec{u}$ of $\vec{v}$

$$= \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

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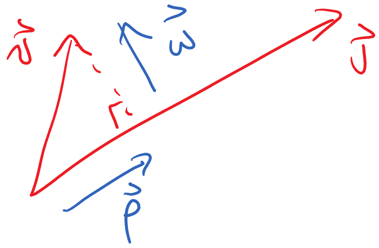
digression : will not be tested



$\vec{v} = \vec{p} + \vec{w}$

$\vec{v}\cdot\vec{u} = (\vec{p}+\vec{w})\cdot\vec{u}$

$$= \vec{p}\cdot\vec{u} + \underbrace{\vec{w}\cdot\vec{u}}_{\text{zero because } \vec{w}\perp\vec{u}}$$

$\vec{v}\cdot\vec{u} = \vec{p}\cdot\vec{u}$

but since $\vec{p}$ is // to $\vec{u}$

then $\vec{p} = c\,\vec{u}$

$$\vec{v}\cdot\vec{u} = c\,\vec{u}\cdot\vec{u}$$

$$c = \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$$

and $p = \dfrac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u}$

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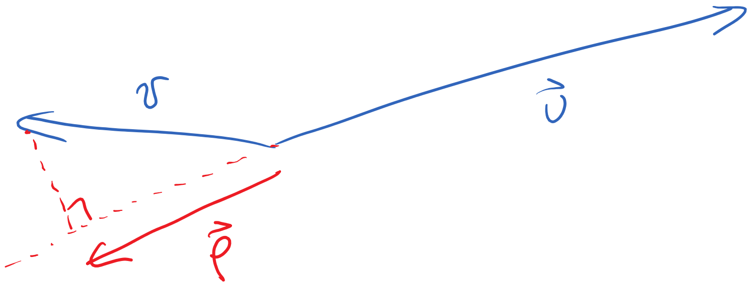
note: if angle between $\vec{u}$ and $\vec{v}$ is greater than 90°, then the projection is in opposite direction to $\vec{u}$



example: find $\text{proj}_{\vec{u}}(\vec{v})$ if

a) $\vec{u} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

answer:

$$\vec{u}\cdot\vec{v} = 4(2) + (-1)(1) + 2(3) = 13$$

$$\vec{u}\cdot\vec{u} = 4^2 + (-1)^2 + 2^2 = 21$$

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

$$u \cdot u$$

$$= \frac{13}{21}\begin{bmatrix}4\\-1\\2\end{bmatrix}$$

b) $\vec{u} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\3\\2\end{bmatrix}$

answer: $\vec{u}\cdot\vec{v} = 3$
$\vec{u}\cdot\vec{u} = 1$

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u}$$

$$= \frac{3}{1}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\3\\0\end{bmatrix}$$

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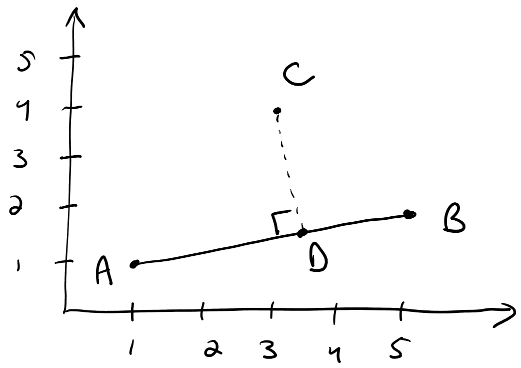
for <u>this</u> example, the projection of $\vec{v}$ onto $\vec{u}$ was the projection of $\vec{v}$ onto the y-axis, so the result is just the y-component

example: Consider the following diagram with $A = (1,1)$, $B = (5,2)$, and $C = (3,4)$. Find point D.



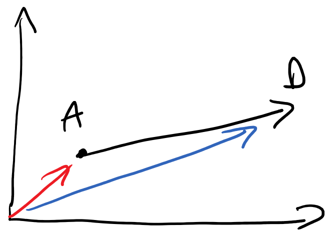
answer:

$\vec{AB} = \begin{bmatrix}4\\1\end{bmatrix}$ $\qquad$ $\vec{AC} = \begin{bmatrix}2\\3\end{bmatrix}$

$$\vec{AD} = \text{proj}_{\vec{AB}}(\vec{AC}) = \frac{\vec{AB}\cdot\vec{AC}}{\vec{AB}\cdot\vec{AB}}\vec{AB}$$

$$= \frac{2\cdot4 + 3\cdot1}{4^2+1^2}\begin{bmatrix}4\\1\end{bmatrix}$$

$$= \frac{11}{17}\begin{bmatrix}4\\1\end{bmatrix}$$



$$\vec{D} = \vec{A} + \vec{AD}$$

$$\vec{OD} = \vec{OA} + \vec{AD}$$

$$= \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}44/17\\11/17\end{bmatrix}$$

$$= \begin{bmatrix}61/17\\28/17\end{bmatrix}$$

the point D is $\left(61/17,\ 28/17\right)$

$\approx (3.59,\ 1.64)$