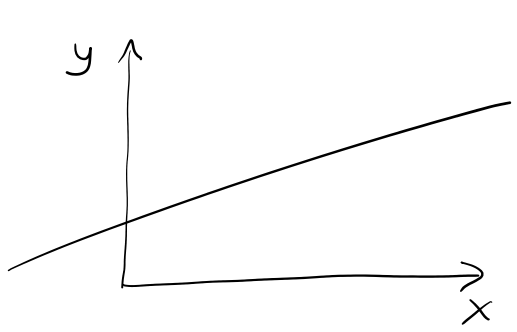
Section 1.3: Lines and Planes

Friday, September 16, 2022 12:56 PM

lines in $\mathbb{R}^2$

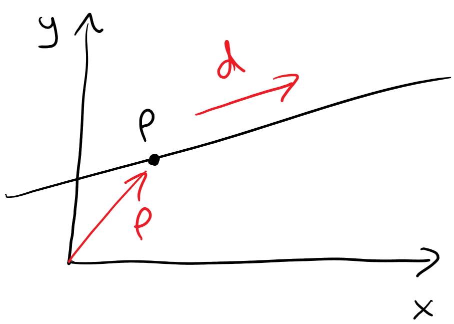
consider a line in $\mathbb{R}^2$:

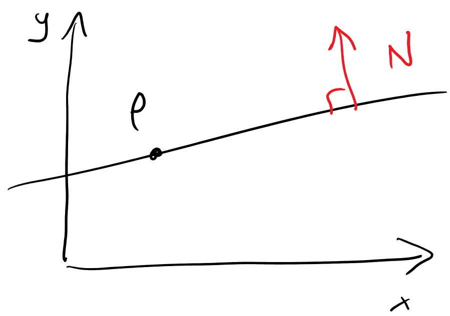


what information do we need to specify this particular line?

- slope and y-intercept (except for vertical lines)
- two points on line (must be different points)

or if we think in terms of vectors





some point P on line and direction vector $\vec{d}$

some point P on line and normal vector $\vec{N}$ where $\vec{N}$ is perpendicular to the line

then all points on the line are found by adding a multiple of

then any point on the line X has

line are found by adding a multiple of $\vec{d}$ to vector $\vec{P}$

then any point on line x has
$$\vec{PX} = \underbrace{t\,\vec{d}}_{\text{some multiple of direction vector}}$$

line X has
$$\vec{PX} \perp \vec{N}$$
$$\vec{PX} \cdot \vec{N} = 0$$

Let's look at
$$\boxed{\begin{gathered}\vec{PX} \cdot \vec{N} = 0 \\ \begin{bmatrix} x - p_1 \\ y - p_2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0\end{gathered}}$$

normal form of line

where $X = (x, y)$
$P = (p_1, p_2)$
$\vec{N} = \begin{bmatrix} a \\ b \end{bmatrix}$

so $\begin{bmatrix} x - 3 \\ y + 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 0$

that is line thru point $P = (-3, 1)$ and normal $N = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

now take the dot product:
$$\boxed{(x - p_1)(a) + (y - p_2)(b) = 0}$$

general equation of a line

so $(x - 3)(2) + (y + 1)5 = 0$
$$\boxed{2x + 5y = 1}$$

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note: $2x + 5y = 1$ general form
$5y = -2x + 1$

$y = -\frac{2}{5}x + \frac{1}{5}$ (slope - intercept form)

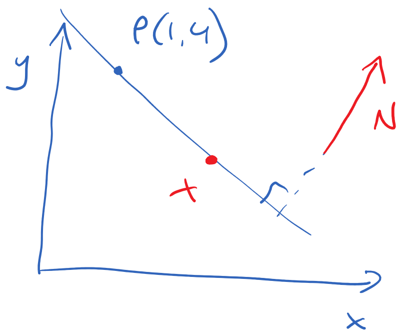
slope $= -\frac{2}{5}$

perpendicular is negative reciprocal:

perpendicular line has slope $\frac{5}{2}$

example: find the general equation of the line perpendicular to $\vec{N} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, passing through the point $P = (1, 4)$.

answer:



let $X = (x, y)$

$\vec{PX} = \begin{bmatrix} x-1 \\ y-4 \end{bmatrix}$

$\vec{PX} \cdot \vec{N} = 0$

$\begin{bmatrix} x-1 \\ y-4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0$

$2(x-1) + 3(y-4) = 0$

$\boxed{2x + 3y = 14}$ general form

note: the general form looks like

$Ax + By = C$ where A, B, C real

and if possible, A, B, and C are integers with A positive you sometimes see

$$Ax + By + D = 0$$

note: short cut for the lazy:

$$P = (1,4)$$

$$\vec{N} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

then general form $\quad Ax + By = C$

$$2x + 3y = C$$

plug in P $\quad 2(1) + 3(4) = C$

$$C = 14$$

$$\boxed{2x + 3y = 14}$$

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in $\mathbb{R}^2$, we'd said that you can define a line by a point P and direction vector $\vec{d}$, where $\vec{d}$ is parallel to the line

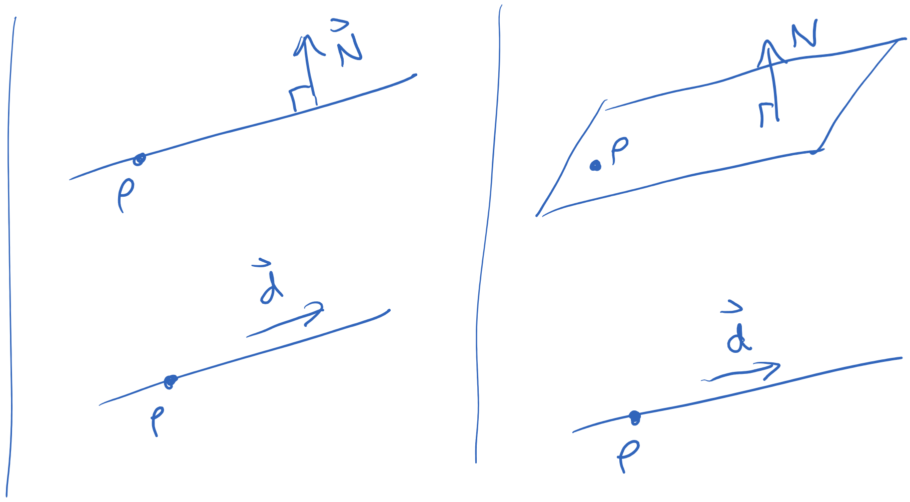
the nice thing is this also works in $\mathbb{R}^3$

$\mathbb{R}^2$ versus $\mathbb{R}^3$:

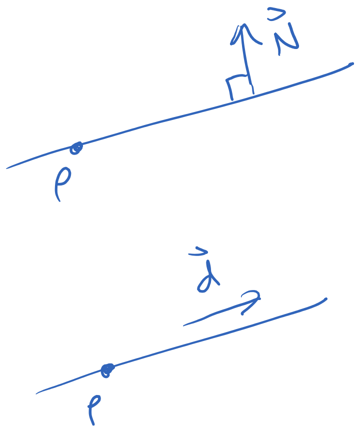
$\mathbb{R}^2$ $\qquad\qquad$ $\mathbb{R}^3$

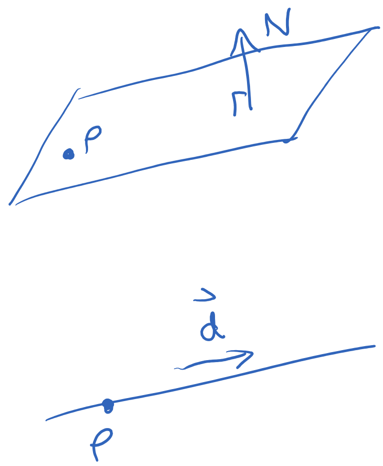
point and normal vector

point and direction vector



point and normal vecter





point and direction vecter

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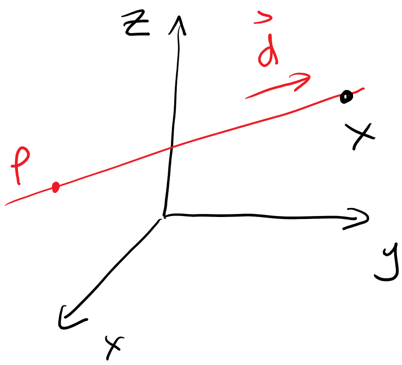
lines in $\mathbb{R}^3$:



given point $P = (p_1, p_2, p_3)$

and direction vecter

$$\vec{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

where $\vec{d}$ is parallel to line

let X be arbitrary point on the line (x, y, z)

then $\vec{PX} \parallel \vec{d}$

$\vec{PX} = t\,\vec{d}$ where t is a scalar

then
$$\boxed{\begin{bmatrix} x - p_1 \\ y - p_2 \\ z - p_3 \end{bmatrix} = t \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}$$
vector equation of line

or
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$
equivalent

$$\begin{bmatrix} y \\ z \end{bmatrix} \quad \begin{bmatrix} p_2 \\ p_3 \end{bmatrix} \quad \begin{bmatrix} d_2 \\ d_3 \end{bmatrix}$$

$$\boxed{\begin{cases} x = p_1 + td_1 \\ y = p_2 + td_2 \\ z = p_3 + td_3 \end{cases}}$$

parametric equations of a line

example: find parametric equations for the line joining points $P = (1, 4, -2)$ and $Q = (2, 3, 5)$.

$$\overrightarrow{PQ} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$$

then $\overrightarrow{PX} = t\,\overrightarrow{PQ}$

$$\begin{bmatrix} x-1 \\ y-4 \\ z+2 \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$$

vector form

$$\boxed{\begin{cases} x = 1 + t \\ y = 4 - t \\ z = -2 + 7t \end{cases}}$$

note: you could use Q as your point and/or $\overrightarrow{QP}$ as your vector

you could also use a multiple of $\overrightarrow{PQ}$

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planes in $\mathbb{R}^3$:

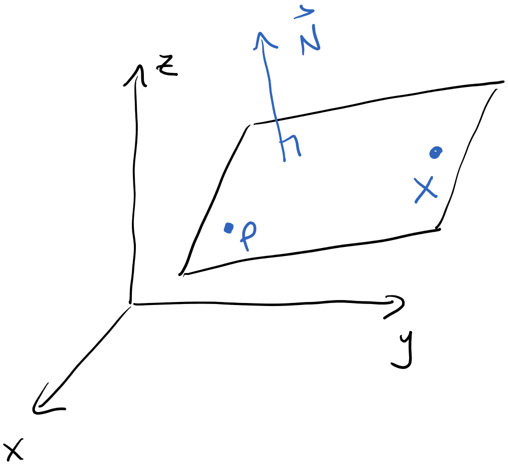
we can describe a plane in $\mathbb{R}^3$ using a point $P = (p_1, p_2, p_3)$ and a vector

$\vec{N} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ perpendicular to plane

$\vec{N} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ perpendicular to plane

[note: in $\mathbb{R}^2$ and $\mathbb{R}^3$, we use the word perpendicular. In $\mathbb{R}^3$ and up, use the word orthogonal]

we want an equation containing arbitrary point $X = (x, y, z)$



so $\vec{PX} \perp \vec{N}$

$$\vec{PX} \cdot \vec{N} = 0$$

$$\boxed{\begin{bmatrix} x - p_1 \\ y - p_2 \\ z - p_3 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0}$$

normal form of a plane

$$n_1(x - p_1) + n_2(y - p_2) + n_3(z - p_3) = 0$$

$$n_1 x + n_2 y + n_3 z = \underbrace{n_1 p_1 + n_2 p_2 + n_3 p_3}_{\text{a constant}}$$

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section 1.3: cont'd 2022/09/21

the general equation for a plane:

$$Ax + By + Cz = D$$

these coefficients are the components of the normal

so $2x + 3y - 5z = 8$ has normal $\vec{N} = \begin{bmatrix} 2 \end{bmatrix}$

so $2x + 3y - 5z = 8$ has normal $\vec{N} = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$

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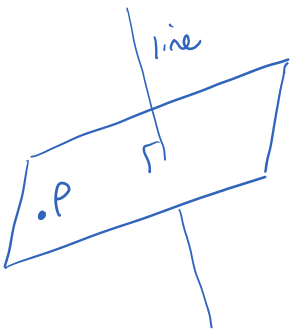
example: Find the equation of a plane through the point $P = (1, 3, 2)$ and perpendicular to the line given by

$$\begin{cases} x = 2 + 3t \\ y = -1 + t \\ z = 2 - 4t \end{cases}$$

answer:



So the line is parallel to the normal

and we can choose any vector from this line to use as the normal

so how do we find this vector?

line:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$$

so $\vec{N} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$

if you don't see that, then find two points on the line and find the vector that joins them

so let $t = 0$, $A = (2, -1, 2)$

$t = 1$, $B = (5, 0, -2)$

$\vec{AB} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$ as before

then $\vec{PX} \cdot \vec{N} = 0$

$$\begin{bmatrix} x-1 \\ y-3 \\ z-2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = 0$$

← normal form for a plane, so could stop here

if the question asked for the general form, then

$$3(x-1) + 1(y-3) - 4(z-2) = 0$$

$$3x + y - 4z = 3 + 3 - 8$$

$$\boxed{3x + y - 4z = -2}$$

general form

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method #2 for the very, very lazy:

point $P = (1, 3, 2)$ and $\vec{N} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$

$$Ax + By + Cz = D$$

$$3x + y - 4z = D$$

plug point in: $3(1) + (3) - 4(2) = D$

$$D = -2$$

$$\boxed{3x + y - 4z = -2}$$

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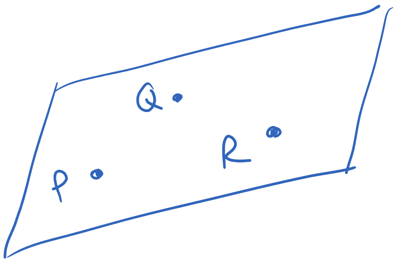
example: Find the general equation of a plane containing the points

$$P = (2, 1, 3)$$
$$Q = (1, 0, 4)$$
$$R = (3, 1, -6)$$

answer:



- need normal and a point

$$\vec{PQ} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad \vec{PR} = \begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 1 \\ 1 & 0 & -9 \end{vmatrix}$$

$$= 9\hat{i} - 8\hat{j} + \hat{k} = \begin{bmatrix} 9 \\ -8 \\ 1 \end{bmatrix}$$

so eqn of plane

$$9x - 8y + z = d$$

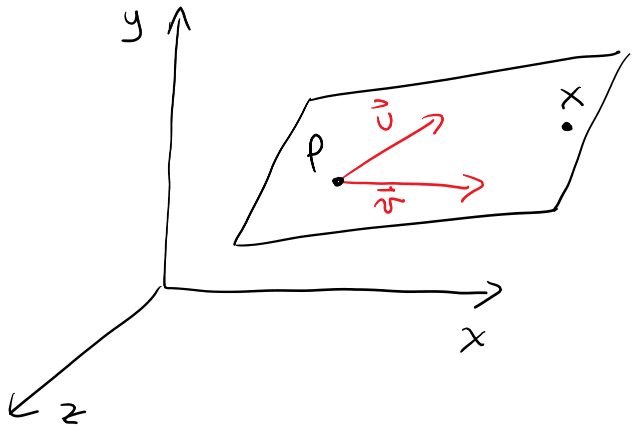
plus in any point: (I'll use P)

$$9(2) - 8(1) + 3 = d$$
$$d = 13$$

$$\boxed{9x - 8y + z = 13}$$

optional check with point $Q = 9(1) - 8(0) + (4) = 13$ ✓
$R = 9(3) - 8(1) - 6 = 13$ ✓

parametric equations for a plane



given a point P in the plane and two vectors $\vec{u}$ and $\vec{v}$ parallel to the plane, where $\vec{u}$ and $\vec{v}$ are not parallel to each other,

this defines a plane

let $X = (x, y, z)$ be some arbitrary point on the plane

then $\vec{PX} = t\vec{u} + s\vec{v}$

$$\boxed{\vec{X} = \vec{P} + t\vec{u} + s\vec{v}}$$

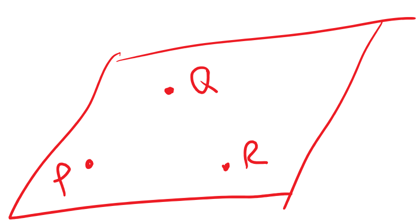
vector form for a plane

the vector from the origin to point X in the plane

if you then write out the individual equations you get the parametric equations

example: Find the parametric equations for the plane containing

$P = (2, 1, 3)$
$Q = (1, 0, 4)$
$R = (3, 1, -6)$



answer: we need a point in the plane and two vectors in the plane

$\vec{p} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ $\vec{PQ} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ $\vec{PR} = \begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}$

then $\vec{x} = \vec{p} + t\,\vec{PQ} + s\,\vec{PR}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}$$ vector form

finally
$$\begin{cases} x = 2 - t + s \\ y = 1 - t \\ z = 3 + t - 9s \end{cases}$$ parametric form

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section 1.3: cont'd 2022/09/22

note: these equations are not unique

any point and any two vectors parallel to the plane will work

note: earlier, we found that the general equation of this plane was

$$9x - 8y + z = 13$$

$$9(2 - t + s) - 8(1 - t) + (3 + t - 9s) = 13$$

$$18 \cancel{-9t} \cancel{+9s} - 8 + \cancel{8t} + 3 \cancel{+t} \cancel{-9s} = 13$$

$$13 = 13 \checkmark$$

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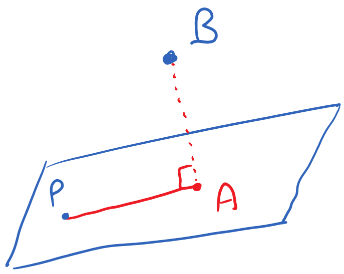
distances: use projections to find the distance from a point to a plane

<u>and</u> a point to a line

example: find the distance between point $B = (3,-1,2)$ and the plane $2x - y + z = 4$.

Also, find the closest point to B in the plane.

answer:



let P be any point in the plane $2x - y + z = 4$

what's an easy way to find a value for P?

set $x=0$, $y=0$ to get $P = (0,0,4)$

let A = point on plane that's closest to B

so distance $d = \|\vec{AB}\|$

but what's $\vec{AB}$? $\quad \vec{AB} = \mathrm{proj}_{\vec{N}}(\vec{PB})$

now $\vec{PB} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$ and $\vec{N} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

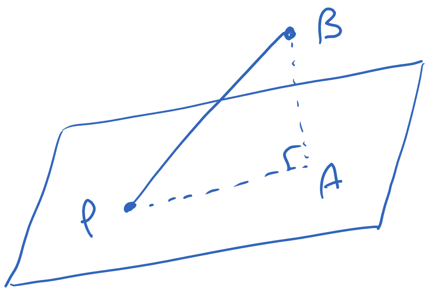
$$\vec{AB} = \mathrm{proj}_{\vec{N}}(\vec{PB}) = \frac{\vec{N}\cdot\vec{PB}}{\vec{N}\cdot\vec{N}}\vec{N}$$

$$= \frac{6+1-2}{4+1+1}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$= \frac{5}{6}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$d = \|\vec{AB}\| = \frac{5}{6}\sqrt{4+1+1} = \boxed{\frac{5}{6}\sqrt{6}}$$

okay, but where is point A?



given P, B, $\vec{PB}$ and $\vec{AB}$

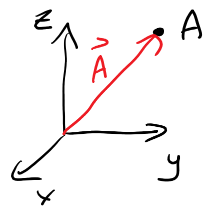
$$\vec{AB} = \vec{B} - \vec{A}$$

so $A = \vec{B} - \vec{AB}$

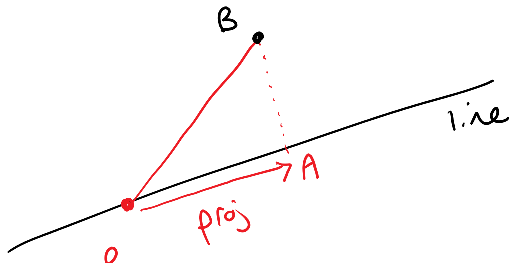
$$= \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \frac{5}{6}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4/3 \\ -1/6 \\ 7/6 \end{bmatrix}$$

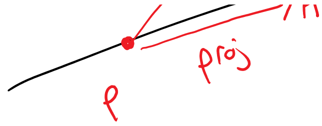
so point $A = \left(\frac{4}{3}, -\frac{1}{6}, \frac{7}{6}\right)$



Distance from a point to a line in $\mathbb{R}^3$



pick some point P on the line

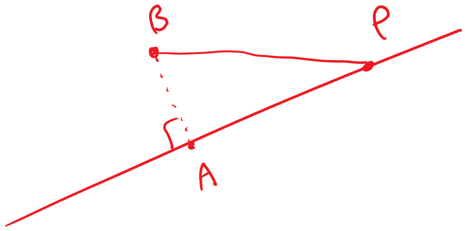


on the line

example: find the distance between the point $B = (1, -1, 2)$ and the line

$$\begin{cases} x = 3 + t \\ y = -2 - 2t \\ z = 4 + 2t \end{cases}$$

answer:



find some point P on the line by letting $t = 0$

$$P = (3, -2, 4)$$

the direction vector $\vec{v}$ along the line is

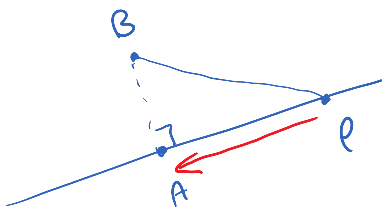
$$\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

then $\vec{PA} = \text{proj}_{\vec{v}}(\vec{PB})$

where $\vec{PB} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$

$$\vec{PA} = \frac{\vec{v} \cdot \vec{PB}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{-2 - 2 - 4}{1 + 4 + 4} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$= -\frac{8}{9} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$



then $\vec{PA} + \vec{AB} = \vec{PB}$

$$\vec{AB} = \vec{PB} - \vec{PA}$$

$$= \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} - \left(-8/9\right) \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -10/9 \\ -7/9 \\ -2/9 \end{bmatrix} = -\frac{1}{9} \begin{bmatrix} 10 \\ 7 \\ 2 \end{bmatrix}$$

$$d = \|\vec{AB}\| = \frac{1}{9}\sqrt{10^2 + 7^2 + 2^2}$$

$$= \frac{1}{9}\sqrt{153} \approx \frac{\sqrt{17}}{3} \approx 1.374$$