Section 2.2: Direct Methods:

Friday, September 23, 2022 1:15 PM

for Solving Linear Systems

definition: A matrix is in row-echelon form if

① any rows with zeros are at the bottom

② in each non-zero row, the first non-zero entry is in a column to the left of any non-zero entries below it

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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section 2.2 cont'd, 2022/09/26

Gaussian elimination - to solve a system of linear equations, transform its augmented matrix to row-echelon form and use back-substitution to solve

example: use Gaussian elimination to solve

$$\begin{cases} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 6y - 5z = 0 \end{cases}$$

answer:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \xrightarrow{} R_2 - 2R_1 \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

$$\downarrow$$

$$R_3 - 3R_1 \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{c} \\ 3R_2 \\ 2R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 6 & -21 & -51 \\ 0 & 6 & -22 & -54 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{c} \\ \\ R_3 - R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 6 & -21 & -51 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{c} \\ \frac{1}{3}R_2 \\ -R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

this corresponds to $\begin{cases} x + y + 2z = 9 \\ 2y - 7z = -17 \\ z = 3 \end{cases}$

now use back substitution: $z = 3$

$$2y - 7z = -17$$
$$2y - 21 = -17$$
$$2y = 4$$
$$y = 2$$

$$x + y + 2z = 9$$
$$x + 2 + 6 = 9$$
$$x = 1$$

solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

definition: leading variables are the variables corresponding to the leading entries in the associated matrix

free variables: all other variables

examples: for the following matrices associated with systems of equations in x, y, and z, give the leading and free variables

a) $$\begin{array}{c}\begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]\end{array}$$

answer: leading : x, y
free : z

b) $$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 8 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

answer:
leading : x, z
free : y

c) $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

answer:
leading : x, y, z
free : none

Gauss-Jordan elimination method:

definition: a matrix is in
RREF
reduced-row echelon form if

① all zero rows are at the bottom

② the leading entry in each non-zero row (leftmost non-zero entry) is a one (called the leading one)

③ each column with a leading one has zeros everywhere else

examples:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

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Gauss-Jordan method:

to solve a system, transform its augmented matrix into RREF and read off the solution

note: in general, we use the following order:

$$\left[\begin{array}{ccc|c} * & \textcircled{6} & \textcircled{5} & \\ \textcircled{1} & * & \textcircled{4} & \\ \textcircled{2} & \textcircled{3} & * & \end{array}\right]$$

non-zero entries

turn the numbered entries into zeros in this order

example: solve

$$\begin{cases} x + y + z = 5 \\ 2x + 3y + 5z = 8 \\ 4x + 5z = 2 \end{cases}$$

answer:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \longrightarrow \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{c} \\ \\ R_3 + 4R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{c} \\ \\ \frac{1}{3} R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{c} R_1 - R_3 \\ R_2 - 3R_3 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\downarrow$$

$$\downarrow$$

$$R_1 - R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$ or $x = 3$, $y = 4$, $z = -2$

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solve the system

$$\begin{cases} x + 2y - 3z = 2 \\ 6x + 3y - 9z = 6 \\ 7x + 14y - 21z = 13 \end{cases}$$

answer: $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right] \xrightarrow[R_3 - 7R_1]{R_2 - 6R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 0 & 0 & 0 & -1 \end{array}\right]$

$$0x + 0y + 0z = -1$$

not possible

$$\boxed{\text{no solution}}$$

(inconsistent system)

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solve

$$\begin{cases} 4y + z = 2 \\ 2x + 6y - 2z = 3 \\ 4x + 8y - 5z = 4 \end{cases}$$

answer: $\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right] \xrightarrow{R_1 \Leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 4 & 8 & -5 & 4 \end{array}\right]$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \end{array}\right.$$

$$\begin{array}{r} \\ \\ R_3 - 2R_1 \end{array}\left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & -4 & -1 & -2 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{r} \\ \frac{1}{4}R_2 \\ R_3 + R_2 \end{array}\left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{r} R_1 - 6R_2 \\ \\ \\ \end{array}\left[\begin{array}{ccc|c} 2 & 0 & -7/2 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{r} \frac{1}{2}R_1 \\ \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 0 & -7/4 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

how to interpret this?

$$\begin{cases} x \quad\quad - 7/4\, z = 0 \\ \quad\quad y + 1/4\, z = 1/2 \end{cases}$$

solve for the leading variables (z is free)

$$\begin{cases} x = \underbrace{7/4\, z}_{} \\ y = \underbrace{-1/4\, z + 1/2}_{\text{the only variables on RHS are free variables}} \end{cases}$$

now, set free variables to parameters

let $z = t$

$$\begin{cases} x = 7/4\, t \\ y = -1/4\, t + 1/2 \\ z = t \end{cases}$$

parametric form

and vector form is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 7/4 \\ -1/4 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix}$$

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example : solve :

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + \cancel{15}^{21} x_6 = \cancel{1}\ 7 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$$

where

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 21 & 7 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \xrightarrow{RREF} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

leading variables x_1, x_3, x_6

free variables x_2, x_4, x_5

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write eqns from RREF:

$$\begin{aligned} x_1 + 3x_2 + 4x_4 + 2x_5 &= 0 \\ x_3 + 2x_4 &= 0 \\ x_6 &= 1/3 \end{aligned}$$

solve for leading variable:

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 1/3 \end{aligned}$$

assign parameters to free variables:

let
$$\begin{aligned} x_2 &= r \\ x_4 &= s \\ x_5 &= t \end{aligned}$$

now write the solution:

$$\begin{cases} x_1 = -3r - 4s - 2t \\ x_2 = r \\ x_3 = -2s \\ x_4 = s \\ x_5 = t \\ x_6 = 1/3 \end{cases}$$

if you need to write it in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

definition - the rank of a matrix is equal to the number of leading ones in the RREF

(= number of leading variables)

in this last example, rank = 3

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example: Given the following two planes in $\mathbb{R}^3$

$$\begin{cases} 4x + y - z = 0 \\ 2x - y + 3z = 4 \end{cases}$$

find the line of intersection.

answer

$$\left[\begin{array}{ccc|c} 4 & 1 & -1 & 0 \\ 2 & -1 & 3 & 4 \end{array}\right] \xrightarrow{RREF} \begin{array}{cccc} x & y & z & \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 2/3 \\ 0 & 1 & -7/3 & -8/3 \end{array}\right]$$

$$x + \tfrac{1}{3} z = \tfrac{2}{3}$$
$$y - \tfrac{7}{3} z = -\tfrac{8}{3}$$

assign parameter to free variable $z = t$

$$\begin{cases} x = -\tfrac{1}{3} t + \tfrac{2}{3} \\ y = \tfrac{7}{3} t - \tfrac{8}{3} \\ z = t \end{cases}$$

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consider the following system:

$$\begin{aligned} x + 2y &= k - 1 \\ 2x + (k^2 - 5)y &= 4 \end{aligned}$$

find the values of k for which the system has

a) a unique solution
b) infinitely many solutions
c) no solutions

answer: $\left[\begin{array}{cc|c} 1 & 2 & k-1 \\ 2 & k^2-5 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & k-1 \\ 0 & k^2-9 & 6-2k \end{array}\right]$

Case 1: $k^2 - 9 = 0$

$k = -3, 3$

if $k = -3$, $\left[\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & 0 & 12 \end{array}\right]$

no solution

if $k = 3$, $\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right]$

infinitely many

Case 2: $k^2 - 9 \neq 0$

$k \neq -3, 3$

$$\left[\begin{array}{cc|c} 1 & 2 & k-1 \\ 0 & k^2-9 & 6-2k \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & k-1 \\ 0 & 1 & \frac{6-2k}{k^2-9} \end{array}\right]$$

$$\frac{6-2k}{k^2-9} = \frac{2(\cancel{3-k})^{-1}}{(\cancel{k-3})(k+3)} = \frac{-2}{k+3}$$

so the matrix in row-echelon form tells us that this will have a single solution

and to answer the original question,

a) $k \neq \pm 3$ $(k \neq -3$ and $k \neq 3)$

b) $k = 3$

c) $k = -3$