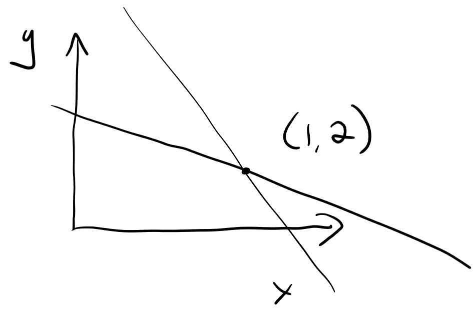
Section 2.3: Spanning Sets and Linear Independence

Tuesday, October 04, 2022 2:09 PM

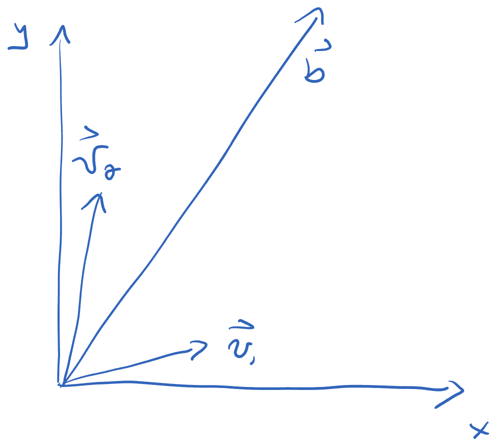
Two geometric interpretations of linear systems:

consider: $\begin{cases} 3x + y = 5 \\ x + 4y = 9 \end{cases}$ has solution $(1, 2)$

row interpretation: two lines



column interpretation: $x \underset{\vec{v}_1}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}} + y \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ 4 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 5 \\ 9 \end{bmatrix}}$



$\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix}$

theorem: a system of linear equations $[A \mid \vec{b}]$ is <u>consistent</u> if and only if $\vec{b}$ is a linear combination of the columns of A

$\hookrightarrow$ unique or infinitely many solutions

Section 2.3: cont'd 2022/10/05

definition: if you have a set of vectors in $\mathbb{R}^n$
$S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots \vec{v}_k\}$,

the set of all linear combinations of those vectors is called the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3 \dots \vec{v}_k$

and is written

$$\text{span}(S) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots \vec{v}_k)$$

example: $\text{span}(\hat{i}, \hat{j}, \hat{k}) = \mathbb{R}^3$

why? because any vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$

can be obtained from a linear combination of $\hat{i}$, $\hat{j}$, and $\hat{k}$

example: is $\begin{bmatrix} 5 \\ -8 \\ -5 \end{bmatrix}$ in the $\text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}\right)$?

answer: in other words, can we find x and y such that

$$x \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + y \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ -5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 1 & -2 & -8 \\ 4 & 1 & -5 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x = -2 \\ y = 3 \end{array}$$

yes, because $-2\begin{bmatrix}2\\1\\4\end{bmatrix} + 3\begin{bmatrix}3\\-2\\1\end{bmatrix} = \begin{bmatrix}5\\-8\\-5\end{bmatrix}$

example: is $\begin{bmatrix}4\\0\\7\end{bmatrix}$ in that same span?

$$\left[\begin{array}{cc|c}2 & 3 & 4\\1 & -2 & 0\\4 & 1 & 7\end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right]$$

$\boxed{\text{No}}$

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example: let $\vec{u} = \begin{bmatrix}2\\1\\3\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}5\\0\\2\end{bmatrix}$

what is span$(\vec{u}, \vec{v})$? give an equation and also describe in words.

answer: $\begin{bmatrix}x\\y\\z\end{bmatrix} = s\begin{bmatrix}2\\1\\3\end{bmatrix} + t\begin{bmatrix}5\\0\\2\end{bmatrix}$

which is a plane in $\mathbb{R}^3$ which contains the origin

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definition: a set of vectors is linearly dependent $\boxed{\text{LD}}$ if there are scalars $c_1, c_2, c_3 \dots c_k$ with at least one scalar non-zero such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \dots c_k\vec{v}_k = \vec{0}$$

otherwise the set is linearly independent $\boxed{\text{LI}}$

otherwise, the set is linearly independent $\boxed{LI}$
and $c_1 = c_2 = c_3 = \dots = c_k$ is the only solution

example: $2\overset{\vec{u}}{\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}} + 3\overset{\vec{v}}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} = \overset{\vec{w}}{\begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}}$

are vectors $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$ LI?

no, they are LD because

$$2\vec{u} + 3\vec{v} - \vec{w} = 0$$

there's at least one non-zero constant that makes this equation true

example: determine if the following sets of vectors are LI or LD. If they are LD, find the relationship between them.

a) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

trivial means all constants are zero

answer: is there a non-trivial solution to

$$c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ ?$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & 4 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

the only solution is $c_1 = c_2 = 0$

$\boxed{LI}$

b) $\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\4\\2\end{bmatrix}$

answer: $\left[\begin{array}{ccc|c}1&1&1&0\\2&1&4&0\\0&-1&2&0\end{array}\right] \xrightarrow{RREF} \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c}1&0&3&0\\0&1&-2&0\\0&0&0&0\end{array}\right]\end{array}$

LD because there's a free variable

$c_1 \quad + 3c_3 = 0$

$c_2 \quad - 2c_3 = 0$

c_3: free variable

Let $c_3 = t$

$$\begin{cases} c_1 = -3t \\ c_2 = 2t \\ c_3 = t \end{cases}$$

or

$$-3t\begin{bmatrix}1\\2\\0\end{bmatrix} + 2t\begin{bmatrix}1\\1\\-1\end{bmatrix} + t\begin{bmatrix}1\\4\\2\end{bmatrix} = 0$$

c) $\begin{bmatrix}2\\1\\3\\5\end{bmatrix}, \begin{bmatrix}3\\2\\4\\1\end{bmatrix}, \begin{bmatrix}3\\1\\7\\1\end{bmatrix}$

$$\left[\begin{array}{ccc|c}2&3&3&0\\1&2&1&0\\3&4&7&0\\5&1&1&0\end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right]$$

$c_1 = c_2 = c_3 = 0$

LI