Section 3.1 : Matrix Operations

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definition : A matrix is a rectangular array of numbers called the "entries" of the matrix

The size of a matrix is $M \times N$ ("M by N") if it has M rows and N columns.

For 3×3 matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

main diagonal

so a_{ij} is in row i and column j

The main diagonal entries of A are $a_{11}, a_{22}, a_{33}, \ldots a_{NN}$

A square matrix has # rows = # columns (is of size $N \times N$)

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① matrix equality

two matrices A and B are equal if and only if they have the same size and $a_{ij} = b_{ij}$ for all i and j

example: $A = \begin{bmatrix} a & 3 \\ 2 & b \end{bmatrix}$ and $B = \begin{bmatrix} 6 & x \\ y & 5 \end{bmatrix}$

if $A = B$, then $a = 6$
$b = 5$

$x = 3$

$y = 2$

example: let $A = \overset{3\times 1}{\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}}$ and $B = \overset{1\times 3}{\begin{bmatrix} 1 & 2 & 5 \end{bmatrix}}$

Does $A = B$? No, not same size.

(2) Matrix Addition/subtraction

example: let $A = \overset{2\times 3}{\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \end{bmatrix}}$ and $B = \overset{2\times 3}{\begin{bmatrix} 3 & 1 & 6 \\ 1 & 2 & 0 \end{bmatrix}}$

Find $A+B$.

$$A+B = \begin{bmatrix} 1+3=4 & 3 & 10 \\ 3 & 5 & 1 \end{bmatrix}$$

definition: $(A+B)_{ij} = a_{ij} + b_{ij}$

$(A-B)_{ij} = a_{ij} - b_{ij}$

note: if A and B are not same size
$A+B$ is undefined
(DNE = does not exist)

(3) Scalar multiplication

if $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \end{bmatrix}$, find $5A$.

five

$$5A = \begin{bmatrix} 5 & 10 & 20 \\ 10 & 15 & 5 \end{bmatrix}$$

definition: $(cA)_{ij} = c\,a_{ij}$

④ Matrix multiplication

AB has size 3×2

same

example: let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times 2}$ and $B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}_{2\times 2}$

Find AB.

$1(7)+2(9)$ $\quad$ $1(8)+2(10)$

answer: $AB = \begin{bmatrix} 25 & 28 \\ 57 & 64 \\ 89 & 100 \end{bmatrix}$

definition: if A is an $M \times N$ matrix and B is an $N \times P$ matrix, then the product AB is defined and has size $M \times P$

$$(AB)_{ij} = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B)$$

dot product

example: now find BA

$2\times 2 \quad 3\times 2$

$B \quad A$

DNE (undefined)

example: $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix}$

a) find AB

$$AB = \begin{bmatrix} 5 & 13 \\ 9 & 31 \end{bmatrix}$$

b) find BA

$$BA = \begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 22 & 15 \\ 18 & 14 \end{bmatrix}$$

In general $AB \neq BA$ if both products are defined

$$\boxed{\text{order matters}}$$

5 Identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Section 3.1: cont'd 2022/10/12

example: let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

a) Find $I_2 A$.

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$$I_2 A = \begin{array}{cc} 2\times 2 & 2\times 3 \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

b) Find $A I_3$.

$$A I_3 = \begin{array}{cc} 2\times 3 & 3\times 3 \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = A$$

⑥ Powers of a Square Matrix

if A is $N \times N$

then $A^2 = AA$

$A^3 = AAA = AA^2$

etc

note: $A^0 = I_N$

warning! don't take short cuts

if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

<u>not</u> $\begin{bmatrix} 1 & 4 \\ 9 & 16 \end{bmatrix}$

⑦ Transpose of a Matrix

example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

definition $(A^T)_{ij} = a_{ji}$

- to get A^T we change the rows of A to columns
- if A is $M \times N$, then $A^T = N \times M$

definition: A square matrix is symmetric iff $A^T = A$

example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ has $A^T = A$

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examples: let $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 4 & 0 \\ 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

a) find A^TC.

$$A^TC = \underset{3\times 2}{\begin{bmatrix} 2 & 4 \\ 3 & 0 \\ 1 & 5 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}} = \underset{3\times 2}{\begin{bmatrix} 4 & 14 \\ 6 & 3 \\ 2 & 16 \end{bmatrix}}$$

b) $AB + C^2$

$$AB + C^2 = \underset{2\times 3}{\begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 5 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & 4 \\ 4 & 0 \\ & \end{bmatrix}} + \underset{2\times 2}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

$$AB+C = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 4 & 0 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 9 \\ 19 & 21 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 14 \\ 19 & 30 \end{bmatrix}$$

c) CB

$$CB = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 4 & 0 \\ 3 & 1 \end{bmatrix} = \text{DNE}$$

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example: let $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$.

find a) AB
b) BA

a) $$AB = \overset{1\times 3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}\overset{3\times 1}{\begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}} = \overset{1\times 1}{\begin{bmatrix} 23 \end{bmatrix}}$$

b) $$BA = \overset{3\times 1}{\begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}}\overset{1\times 3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} = \overset{3\times 3}{\begin{bmatrix} 3 & 6 & 9 \\ 1 & 2 & 3 \\ 6 & 12 & 18 \end{bmatrix}}$$

note: (can skip) there is another way to multiply a matrix by a column vector

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 17 \\ 41 \end{bmatrix}$$

we'll skip block multiplication