Section 3.3 : The Inverse of a Matrix

Thursday, October 13, 2022 10:07 AM

definition : the inverse of an $N \times N$ matrix A is an $N \times N$ matrix A^{-1} such that

$$AA^{-1} = I_N$$

$$A^{-1}A = I_N$$

note: matrix must be square to have an inverse

but not all square matrices have an inverse (however, if it does have an inverse, that inverse is unique)

special case: 2×2 matrices

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

the determinant of A is

$$\det(A) = ad - bc$$

then $A^{-1} = \dfrac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

swap main diagonal, other entries negate

note: A^{-1} only exists if $\det(A) \neq 0$

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example: find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

answer: $\det(A) = ad - bc = 1(4) - 2(3) = -2$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

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check: $AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$A^{-1}A$ is similarly equal to I_2

why do we care?

suppose $AB = C$ where A, B, and C are matrices and A are C are known - we want to find B

long, horrible method #1:
DON'T DO THIS!

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 12 & 9 \end{bmatrix}$$

then
$$\left.\begin{aligned} a + 2c &= 4 \\ b + 2d &= 5 \\ 3a + 4c &= 12 \\ 3b + 4d &= 9 \end{aligned}\right\}$$
4×4 system that I need to solve

YIKES!

or somewhat less annoying method #2:

$AB = C$ where $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$

$AB = C$ where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$C = \begin{bmatrix} 4 & 5 \\ 12 & 9 \end{bmatrix}$

find B

answer: $AB = C$

$A^{-1}(AB) = A^{-1}C$

$(A^{-1}A)B = A^{-1}C$

$I_2 B = A^{-1}C$

$B = A^{-1}C$

now $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ so $A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

so $B = A^{-1} C$

$= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 12 & 9 \end{bmatrix}$

$= \begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix}$

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example: solve the following system using the matrix inverse

$$\begin{cases} 3x + 2y = 1 \\ 6x - y = 17 \end{cases}$$

answer: this gives the matrix equation

$$\begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}$$

$$A \quad \vec{x} = \vec{b}$$

now solve for $\vec{x}$:

$$\begin{aligned} A\vec{x} &= \vec{b} \\ A^{-1}A\vec{x} &= A^{-1}\vec{b} \\ I\vec{x} &= A^{-1}\vec{b} \\ \vec{x} &= A^{-1}\vec{b} \end{aligned}$$

$\Leftarrow$ can just write this

$$\det(A) = 3(-1) - 2(6) = -15$$

$$A^{-1} = \frac{1}{-15}\begin{bmatrix} -1 & -2 \\ -6 & 3 \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} 1 & 2 \\ 6 & -3 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b} = \frac{1}{15}\begin{bmatrix} 1 & 2 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 17 \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} 35 \\ -45 \end{bmatrix} = \begin{bmatrix} 7/3 \\ -3 \end{bmatrix} \text{ or } \frac{1}{3}\begin{bmatrix} 7 \\ -9 \end{bmatrix}$$

optional check: $3x + 2y = 1$

$$3(7/3) + 2(-3) = 1 \quad \checkmark$$

$$6x - y = 17$$

$$6(7/3) - (-3) = 17 \quad \checkmark$$

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see handout for Properties of the Inverse of a Matrix

warning! the one we need to watch out for is

$$(AB)^{-1} = B^{-1}A^{-1}$$

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my tablet died (stopped recognizing pen input) so I went to the board. Below I have copied the relevent notes from Fall 2021:

why?

$$AB\,(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$
$$= A\,I\,A^{-1}$$
$$= AA^{-1}$$
$$= I$$

$$(B^{-1}A^{-1})\,AB = B^{-1}(A^{-1}A)\,B$$
$$= B^{-1}\,I\,B$$
$$= B^{-1}B$$
$$= I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

For matrices larger than 2×2, we use Gauss-Jordan method to find A^{-1}:

$$[A \mid I] \xrightarrow{RREF} [I \mid A^{-1}]$$

example: let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -5 \\ -1 & -2 & 0 \end{bmatrix}$

a) Find A^{-1}.

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$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 7 & -5 & 0 & 1 & 0 \\ -1 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-3R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1-2R_2 \\ \\ -R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 7 & -2 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right]$$

$$\begin{array}{c} R_1-3R_3 \\ R_2+2R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -2 & 3 \\ 0 & 1 & 0 & -5 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 10 & -2 & 3 \\ -5 & 1 & -2 \\ -1 & 0 & -1 \end{bmatrix}}_{A^{-1}}$$

check: $AA^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -5 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 10 & -2 & 3 \\ -5 & 1 & -2 \\ -1 & 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) Use A^{-1} to solve $\begin{cases} x + 2y - z = 2 \\ 3x + 7y - 5z = 5 \\ -x - 2y = 1 \end{cases}$

$$\underbrace{\overset{A}{\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -5 \\ -1 & -2 & 0 \end{bmatrix}}}_{\text{same matrix from part (a)}} \overset{\vec{x}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \overset{\vec{b}}{\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}}$$

$$A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = \begin{bmatrix} 10 & -2 & 3 \\ -5 & 1 & -2 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

$$= \begin{bmatrix} 13 \\ -7 \\ -3 \end{bmatrix}$$

$$\boxed{x = 13, \quad y = -7, \quad z = -3}$$

but why does the Gauss Jordan method work?

definition: an <u>elementary matrix</u> is a matrix that can be obtained by performing one elementary row operation on an identity matrix

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big idea for elementary matrices:

$$A \xrightarrow[(R_2 - 2R_1)]{\text{some row op}} B$$

$$I \xrightarrow[(R_2 - 2R_1)]{\text{exactly the same row op}} E$$

then $B = EA$

note: A, B do <u>not</u> have to be square

example:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 6 & 8 \end{bmatrix} \xrightarrow{R_2 \Leftrightarrow R_3} B = \begin{bmatrix} 2 & 3 \\ 6 & 8 \\ 4 & 1 \end{bmatrix}$$

then
$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \Leftrightarrow R_3} E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and we note that

3×3 $\quad$ 3×2

and we note that

$$EA = \overset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \overset{3\times 2}{\begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 6 & 8 \end{bmatrix}} = \begin{bmatrix} 2 & 3 \\ 6 & 8 \\ 4 & 1 \end{bmatrix}$$

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example:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 - 4R_1} B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -11 & -5 \\ 2 & 1 & 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - 4R_1} E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and if you want, you could check that $EA = B$

recall: we only have 3 elementary row ops:

1. $R_i \Leftrightarrow R_j$
2. cR_i where $c \neq 0$
3. replace R_i by $R_i + cR_j$

elementary matrices are <u>invertible</u>:

$$E E^{-1} = I$$

↑ undoes the elementary row op of E

examples:

a) $$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{} = I \quad \xrightarrow{6R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E$$

what is E^{-1}?

the inverse matrix E^{-1} just undoes that row op

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\frac{1}{6}R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

totally optional check:

$$EE^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \Leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E$$

what's E^{-1}?

$$E^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so if we get from A to B using k elementary row operations

$$A \xrightarrow{k \text{ row ops}} B$$

then $B = E_k E_{k-1} \ldots\ldots E_2 E_1 A$

$\uparrow$ corresponds to the first row op

proof of Gauss-Jordan method for inverses

if $A \xrightarrow{k \text{ row ops}} I$

then $I = \underbrace{E_k E_{k-1} E_{k-2} \ldots\ldots E_2 E_1}\, A$

then $I = \underbrace{E_k E_{k-1} E_{k-2} \cdots E_2 E_1}_{A^{-1}} A$

so $A^{-1} = E_k E_{k-1} E_{k-2} \cdots E_2 E_1 I$

that's why $[A \mid I] \xrightarrow{k \text{ row ops}} [I \mid A^{-1}]$

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example: let $A = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$. Express A and A^{-1} as a product of elementary matrices

answer:

row operations

$$[A \mid I] = \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

$\downarrow R_1 \leftrightarrow R_2$

$$\left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right]$$

$\downarrow R_2 - 2R_1$

$$\left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -3 & 1 & -2 \end{array}\right]$$

$\downarrow R_1 + R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & -3 & 1 & -2 \end{array}\right]$$

$\downarrow -\tfrac{1}{3} R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -\tfrac{1}{3} & \tfrac{2}{3} \end{array}\right]$$

associated elementary matrices

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\downarrow R_1 \leftrightarrow R_2$

$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$R_2 - 2R_1$

$E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$R_1 + R_2$

$E_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$-\tfrac{1}{3} R_2$

$E_4 = \begin{bmatrix} 1 & 0 \\ 0 & -\tfrac{1}{3} \end{bmatrix}$

opposite row ops (inverses)

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\downarrow R_1 \leftrightarrow R_2$

$E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$R_2 + 2R_1$

$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$R_1 - R_2$

$E_3^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$E_4^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$

then $I = \underbrace{E_4 E_3 E_2 E_1}_{A^{-1}} A$

so $A^{-1} = E_4 E_3 E_2 E_1$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $A = (A^{-1})^{-1}$

$$= (E_4 E_3 E_2 E_1)^{-1}$$

$$= E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$

note: these matrices are not unique since you could have used different row ops

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example: Find A if

$$(I_2 + 2A)^{-1} = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$$

$$I_2 + 2A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}^{-1}$$

$$I_2 + 2A = \frac{1}{2(4) - 6(1)} \begin{bmatrix} 4 & -6 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 4 & -6 \\ -1 & 2 \end{bmatrix}$$

$$2A = \frac{1}{2} \begin{bmatrix} 4 & -6 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$A = \frac{1}{2} \begin{bmatrix} 1 & -3 \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$A = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{4} & 0 \end{bmatrix}$$

example: solve for X, assuming all matrices are invertible

$(B^{-1})^2$

$$B^2 A^2 X A^{-1} B^{-1} = I$$

$$\underbrace{B^{-2} B^2}_{I} A^2 X A^{-1} B^{-1} = B^{-2} I$$

$$\underbrace{A^{-2} A^2}_{I} X A^{-1} B^{-1} = A^{-2} B^{-2} I$$

$$X A^{-1} \underbrace{B^{-1} B} = A^{-2} B^{-2} B$$

$$X A^{-1} = A^{-2} B^{-1} \underbrace{B^{-1} B}_{I}$$

$$X A^{-1} A = A^{-2} B^{-1} A$$

$$\boxed{X = A^{-2} B^{-1} A}$$

optional check:

$$B^2 A^2 X A^{-1} B^{-1} = I$$

$$B^2 \underbrace{A^2 A^{-2}} B^{-1} \underbrace{A A^{-1}} B^{-1} = I$$

$$B^2 B^{-1} B^{-1} = I \quad \checkmark$$