Section 3.4: The LU Factorization

Thursday, October 20, 2022 9:55 AM

$$\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 2 & -1 & 6 \end{bmatrix}$$ lower triangular matrix

non-zero entries highlighted

$$\begin{bmatrix} 2 & 6 & 1 \\ 0 & 3 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$ upper triangular matrix

for a linear system

$$A\vec{x} = \vec{b}$$

where A is a square matrix, we want to express A as

$$A = LU$$

where L = lower triangular matrix

U = upper triangular matrix

note: L and U are not unique

note: textbook goes further and insists that L be unit triangular

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

ones on main diagonal

then to solve for $\vec{x}$:

$$A\vec{x} = \vec{b}$$

$$LU\vec{x} = \vec{b}$$

$$L(U\vec{x}) = \vec{b}$$

call $U\vec{x} = \vec{y}$

$L\vec{y} = \tilde{b}$

why is this cool?

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 13 \end{bmatrix}$$

having lower triangular matrix means you can substitute to find $\vec{y}$ easily

then do $U\vec{x} = \vec{y}$

you just found this

$$\begin{bmatrix} 8 & 3 & 9 \\ 0 & 5 & 2 \\ 0 & 0 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

and solve for $\vec{x}$

---

example: Find the LU factorization of A where

$$A = \begin{bmatrix} 10 & -4 & 0 \\ -10 & 12 & -6 \\ 5 & 14 & -10 \end{bmatrix}$$

and use it to solve

$$\begin{cases} 10x - 4y = 16 \\ -10x + 12y - 6z = -2 \\ 5x + 14y - 10z = 34 \end{cases}$$

answer:

step 1: transform A into U and keep track of corresponding elementary row ops and matrices

- use only $R_i + cR_j$ and cR_i } where R_i is <u>above</u> R_j

but <u>not</u> rows swaps

Section 3.4 cont'd 2022/10/21

$$A = \begin{bmatrix} 10 & -4 & 0 \\ -10 & 12 & -6 \\ 5 & 14 & -10 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\downarrow R_2 + R_1$ $\qquad$ $\downarrow R_2 + R_1$ $\qquad$ $R_2 - R_1 \downarrow$

$$\begin{bmatrix} 10 & -4 & 0 \\ 0 & 8 & -6 \\ 5 & 14 & -10 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\downarrow R_3 - \frac{1}{2}R_1$ $\qquad$ $R_3 - \frac{1}{2}R_1$ $\qquad$ $R_3 + \frac{1}{2}R_1$

$$\begin{bmatrix} 10 & -4 & 0 \\ 0 & 8 & -6 \\ 0 & 16 & -10 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$$

$\downarrow R_3 - 2R_2$ $\qquad$ $R_3 - 2R_2$ $\qquad$ $R_3 + 2R_2$

$$U = \begin{bmatrix} 10 & -4 & 0 \\ 0 & 8 & -6 \\ 0 & 0 & 2 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 A = U$$

$$E_1^{-1} E_2^{-1} \underbrace{E_3^{-1} E_3} E_2 E_1 A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

$$A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}}_{L} U$$

$$L = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix}$$

optional check:

optional check:

$$\boxed{LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix} \begin{bmatrix} 10 & -4 & 0 \\ 0 & 8 & 6 \\ 0 & 0 & 2 \end{bmatrix}} = \begin{bmatrix} 10 & -4 & 0 \\ -10 & 12 & -6 \\ 5 & 14 & -10 \end{bmatrix} = A$$

b) solve $\quad A \overset{\vec{x}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \overset{\vec{b}}{\begin{bmatrix} 16 \\ -2 \\ 34 \end{bmatrix}}$

answer: $\quad A\vec{x} = \vec{b}$

$$L\underbrace{U\vec{x}}_{\vec{y}} = \vec{b}$$

$$L\vec{y} = \vec{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 16 \\ -2 \\ 34 \end{bmatrix}$$

so $y_1 = 16$

$-y_1 + y_2 = -2$ $\qquad -16 + y_2 = -2$ and $y_2 = 14$

$\frac{1}{2} y_1 + 2y_2 + y_3 = 34$ $\qquad \frac{1}{2}(16) + 2(14) + y_3 = 34$ so $y_3 = -2$

but remember $\quad \vec{y} = U\vec{x}$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 & -4 & 0 \\ 0 & 8 & -6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 16 \\ 14 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 & -4 & 0 \\ 0 & 8 & -6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

then $16 = 10x - 4y$ $\qquad 16 = 10x - 4(1)$ $\qquad$ so $x = 2$

$14 = 8y - 6z$ $\qquad 14 = 8y - 6(-1)$ $\qquad$ so $y = 1$

$-2 = 2z$ $\qquad$ so $z = -1$

so $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

note: to solve 3×3 systems, Gauss-Jordan is more efficient. But for large systems, LU factorization is more efficient

---

happy, fun shortcut

- you must be able to reduce A to row-eschelon form (U) without swapping any rows

$\Rightarrow$ only $R_i + cR_j$

↑ j is a row above R_i

(if you need to swap rows, need something called a permutation matrix — in textbook but we don't cover it)

example: find an LU factorization of

$$A = \begin{bmatrix} -3 & 1 & 2 \\ 6 & 2 & -5 \\ 9 & 5 & -6 \end{bmatrix}$$

answer: $A = \begin{bmatrix} -3 & 1 & 2 \\ 6 & 2 & -5 \\ 9 & 5 & -6 \end{bmatrix}$

$\downarrow R_2 + 2R_1$

$$\begin{bmatrix} -3 & 1 & 2 \\ 0 & 4 & -1 \\ 9 & 5 & -6 \end{bmatrix}$$

$\downarrow R_3 + 3R_1$

$$\begin{bmatrix} -3 & 1 & 2 \\ 0 & 4 & -1 \\ 0 & 8 & 0 \end{bmatrix}$$

$\downarrow R_3 - 2R_2$

$$L = \begin{array}{cl} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix} & \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1,\ R_3 + 2R_2 \end{array} \end{array}$$

$$\begin{bmatrix} 0 & 8 & 0 \end{bmatrix}$$

$$\downarrow R_3 - 2R_2$$

$$U = \begin{bmatrix} -3 & 1 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

example: solve the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ -3 \\ -1 \\ 0 \end{bmatrix}$$

answer:

$$A\vec{x} = \vec{b}$$

$$L\underbrace{U\vec{x}}_{\vec{y}} = \vec{b}$$

$$L\vec{y} = \vec{b} \qquad \text{solve for } \vec{y}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -1 \\ 0 \end{bmatrix}$$

so $y_1 = 1$

$-2y_1 + y_2 = -3$ so $-2 + y_2 = -3$ and $y_2 = -1$

$3y_1 - 2y_2 + y_3 = -1$ so $3 + 2 + y_3 = -1$ and $y_3 = -6$

$-5y_1 + 4y_2 - 2y_3 + y_4 = 0$ so $-5 - 4 + 12 + y_4 = 0$ and $y_4 = -3$

then $U\vec{x} = \vec{y}$

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -6 \\ -3 \end{bmatrix}$$

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 1 \\ 3x_2 + 5x_3 + 2x_4 &= -1 \\ -2x_3 &= -6 \\ x_4 &= -3 \end{aligned}$$

$x_1 - 40/3 + 9 = 1$ so $x_1 = 16/3$

$3x_2 + 15 - 6 = -1$ so $x_2 = -10/3$

$x_3 = 3$

so $x_4 = -3$

$$\begin{aligned} x_1 &= 1 - 9 + 40/3 \\ &= \frac{3}{3} - \frac{27}{3} + \frac{40}{3} \\ &= 16/3 \end{aligned}$$

so $\vec{x} = \begin{bmatrix} 16/3 \\ -10/3 \\ 3 \\ -3 \end{bmatrix}$