Section 3.5: Subspaces, Basis, Dimension, and Rank

Monday, October 24, 2022 9:50 AM

definition: a subspace of $\mathbb{R}^n$ is a collection of vectors S in $\mathbb{R}^n$ such that

1) $\vec{0}$ (the zero vector) is in S

2) if $\vec{u}$ and $\vec{v}$ are in S, then $(\vec{u} + \vec{v})$ is in S

3) if $\vec{u}$ is in S and c is any scalar, then $c\vec{u}$ is also in S

example: in $\mathbb{R}^3$, any line or plane containing the origin is a subspace

example: consider the set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x = 2y$ and $z = -4y$. Is this a subspace of $\mathbb{R}^3$?

answer: let $y = t$ (no restrictions on y)

then $x = 2t$
$y = t$
$z = -4t$

and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$

now look at the three conditions

1) is $\vec{0}$ in the span? yes, when $t = 0$

2) suppose $\begin{bmatrix} 2a \\ a \\ -4a \end{bmatrix}$ and $\begin{bmatrix} 2b \\ b \\ -4b \end{bmatrix}$ are in

the span. Is $\begin{bmatrix} 2a \\ \end{bmatrix} + \begin{bmatrix} 2b \\ \end{bmatrix}$ also in the span. Is $\begin{bmatrix} 2a \\ a \\ -4a \end{bmatrix} + \begin{bmatrix} 2b \\ b \\ -4b \end{bmatrix}$ also in the span?

$$\begin{bmatrix} 2a \\ a \\ -4a \end{bmatrix} + \begin{bmatrix} 2b \\ b \\ -4b \end{bmatrix} = \begin{bmatrix} 2a+2b \\ a+b \\ -4a-4b \end{bmatrix} = (a+b)\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

multiple of $\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$ so yes

example: consider the set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x = 2y+1$ and $z = -4y$.

Is this set a subspace?

answer: let $y = t$ $\begin{cases} x = 2t+1 \\ y = t \\ z = -4t \end{cases}$

1) is $\vec{0}$ in this set? no! there is no value of t that simultaneously makes all three components zero

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Given any matrix A, there are three fundamental spaces associated with it

(1) the row space

(2) the column space

(3) the null space

example: let $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 4 & 3 \end{bmatrix}$

what is $Row(A)$? $Col(A)$?

answer: $Row(A)$ = the set of all vectors we can make from linear combinations of the rows of A

$= span\left(\begin{bmatrix} 1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \end{bmatrix} \right)$

note: this is a subspace of $\mathbb{R}^2$

$Col(A) = span\left(\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right)$

note: this is a subspace of $\mathbb{R}^3$, and is in fact a plane through the origin

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definition: let A be an $M \times N$ matrix

① The row space of A is the subspace of $\mathbb{R}^N$ spanned by the rows of A. It is denoted by

$Row(A)$.

② The column space of A is the subspace of $\mathbb{R}^M$ spanned by the columns of A.

It is denoted by

Col(A)

example: let $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 2 & -4 \end{bmatrix}$

a) Is $\vec{b} = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$ in Col(A)?

- is $\vec{b}$ in the span of these vectors? (columns)?
- is $\vec{b}$ a linear combination of these columns?

$$s\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 1 & -1 \\ 2 & -4 & 8 \end{array}\right] \xrightarrow{\text{RREF}} \begin{array}{ccc} s & t & \\ \end{array}\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

so $b = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix}$

$\boxed{\text{yes}}$

b) is $\vec{w} = \begin{bmatrix} 3 & 2 \end{bmatrix}$ in Row(A)? $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 2 & -4 \end{bmatrix}$

method #1: (not recommended)

you could, if you insist, solve

you could, if you insist, solve

$$\left[\begin{array}{cc} 1 & -2 \\ 0 & 1 \\ 2 & -4 \\ \hline 3 & 2 \end{array}\right]$$

this augmented matrix using column operations (not row ops) to get into RCEF

column

method #2: take the transpose $[A^T \mid \vec{w}^T]$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ -2 & 1 & -4 & 2 \end{array}\right]$$

$$\downarrow R_2 + 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 8 \end{array}\right]$$

free variable, infinitely many solutions

$\boxed{\text{yes}}$

theorem: if 2 matrizes are row-equivalent, then they have the same row space

so if $A \xrightarrow{\text{row ops}} B$

then $Row(A) = Row(B)$

example: $A = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}$

give $Row(A)$ and $Col(A)$.

answer: $Row(A) = span\left([1 \;\; 5], [2 \;\; 10] \right)$

this second vector is a multiple of the first

so can write

$\text{Row}(A) = \text{span}([1 \quad 5])$

$\text{Col}(A) = \text{span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}5\\10\end{bmatrix}\right)$

or just $\text{span}\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$

but note

$$A = \begin{bmatrix}1 & 5\\2 & 10\end{bmatrix} \xrightarrow[R_2 - 2R_1]{} B = \begin{bmatrix}1 & 5\\0 & 0\end{bmatrix}$$

note $\text{Row}(B) = \text{span}([1 \quad 5])$

BUT $\text{Col}(B) = \text{span}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}5\\0\end{bmatrix}\right)$

$\neq \text{Col}(A)$

conclusion: row operations change the column space

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definition: a basis of a subspace S of $\mathbb{R}^n$ is a set of vectors in S such that

1) the span of the vectors in the basis is S

2) the vectors in the basis are LI

note: I like to think of a basis as the minimum set of vectors needed to span the space

example: the set of vectors $\{\hat{i}, \hat{j}, \hat{k}\}$ is a basis of $\mathbb{R}^3$

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notation

$\{\ \}$ is used for a set

$\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right\}$ is a set of vectors

a set is a collection of objects (numbers, matrices, vectors)

$\underbrace{\text{span}(\ )}_{\text{function notation } f(x)}$

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definition: the dimension of a subspace S, where S is not the zero vector, is the number of vectors in a basis for S

if S is the zero vector, then the dimension of S is zero examples: the subspaces of $\mathbb{R}^3$:

1) the point $\vec{0}$ has dimension 0

2) any line through the origin has dimension 1

3) any plane through the origin has dimension 2

4) the entire space $\mathbb{R}^3$ is a volume with dimension 3

definition: $\text{Null}(A) = \{ \vec{x} \text{ in } \mathbb{R}^N \mid A\vec{x} = \vec{0} \}$

the null space of $M \times N$ matrix A is the set of all vectors in $\mathbb{R}^N$ such that when those vectors are left-multiplied by A, the result is the zero vector $\vec{0}$

note: row ops on A do not change the null space

example: find a basis for the row space, the column space, and the null space of

$$A = \begin{bmatrix} 3 & 7 & 4 & 2 & 0 \\ 4 & 3 & -1 & 3 & 1 \\ 4 & 8 & 4 & 5 & 7 \\ 5 & 4 & -1 & 6 & 8 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

answer: row space

$\text{Row}(A) = \text{Row}(\text{RREF})$

$= \text{span}$ (non-zero rows of RREF)

so the basis for $\text{Row}(A)$ is

$$\{[1\ 0\ -1\ 0\ -2], [0\ 1\ 1\ 0\ 0], [0\ 0\ 0\ 1\ 3]\}$$

curly brackets means the set of these three vectors

Column space

- recall that row ops change the column space, so we can't just read off of the RREF

- but the RREF will tell us which columns in original matrix to use

Col(A) = span(columns of A with leading ones in the RREF)

there are leading ones in columns 1, 2, and 4

so basis of Col(A) is

$$\left\{ \begin{bmatrix} 3 \\ 4 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \\ 6 \end{bmatrix} \right\}$$

note: column 3 and column 5 were not included because they are linear combinations of the other columns

- the third column of the RREF is a recipe for column 3:

$$\vec{C}_3 = -1\, C_1 + 1\, \vec{C}_2$$

similarly

$$C_5 = -2C_1 + 3C_4$$

null space:

$$\text{Null}(A) = \{\vec{x} \mid A\vec{x} = 0\}$$

the set of all $\vec{x}$ such that $A\vec{x} = 0$

$$A\vec{x} = \underbrace{\begin{bmatrix} 3 & 7 & 4 & 2 & 0 \\ 4 & 3 & -1 & 3 & 1 \\ 4 & 8 & 4 & 5 & 7 \\ 5 & 4 & -1 & 6 & 8 \end{bmatrix}}_{4\times 5} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{5\times 1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{4\times 1}$$

$$[\vec{A} \mid \vec{0}] \xrightarrow{\text{row ops}} [\text{RREF} \mid \vec{0}]$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free variables

assign parameters to free variables let $x_3 = s$, $x_5 = t$

$$\begin{cases} x_1 - x_3 - 2x_5 = 0 \\ x_2 + x_3 = 0 \\ x_4 + 3x_5 = 0 \end{cases}$$

$$\begin{cases} x_1 = s + 2t \\ x_2 = -s \\ x_3 = s \\ x_4 = -3t \\ x_5 = t \end{cases}$$

$$\vec{x} = s\begin{bmatrix}1\\-1\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}2\\0\\0\\-3\\1\end{bmatrix}$$

these two vectors are the basis for Null(A)

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quick conceptual example:

there is a light source directly overhead that will cast the shadow of your fingertip onto a desk

let $\vec{x} = \begin{bmatrix}x\\y\\z\end{bmatrix}$ be the position of your fingertip in space

then $\vec{y} = \begin{bmatrix}x\\y\\0\end{bmatrix}$ is the position of your fingertip's shadow on the desk

and the projection onto the desk (position of the shadow) is

$$\vec{y} = A\vec{x}$$

where $A = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ note: this is in RREF already

question: give a basis for the row space and the null space of A

answer: row space: $\{[1\ \ 0\ \ 0], [0\ \ 1\ \ 0]\}$

null space:

$$A\vec{x} = \vec{0}$$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

free variable is z

assign parameter: let $z = t$

$$\begin{cases} x = 0 \\ y = 0 \\ z = t \end{cases} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

basis for null space

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in general,

Rank(A) = dimension of Row(A)
- number of vectors in Row(A)

= dimension of Col(A)
- number of vectors in Col(A)

= number of leading ones in RREF

Nullity(A) = dimension of Null(A)
- number of vectors in Null(A)

= number of free variables in RREF

recall: every column in RREF has either a leading or free variable

theorem (Rank - Nullity)

if A is $M \times N$

then N is the number of variables

$N = \#\text{leading} + \#\text{free}$

$= \text{Rank}(A) + \text{Nullity}(A)$

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example: find a basis of the vectors

$S = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$

in $\mathbb{R}^5$ where

$\vec{v}_1 = [1 \quad -2 \quad -3 \quad 2 \quad -4]$,

$\vec{v}_2 = [-3 \quad 7 \quad -1 \quad 1 \quad -3]$,

$\vec{v}_3 = [2 \quad -5 \quad 4 \quad 3 \quad 7]$,

$\vec{v}_4 = [-3 \quad 6 \quad 9 \quad -6 \quad 1]$,

note: $A = \begin{bmatrix} 1 & -2 & -3 & 2 & -4 \\ -3 & 7 & -1 & 1 & -3 \\ 2 & -5 & 4 & 3 & 7 \\ -3 & 6 & 9 & -6 & 1 \end{bmatrix}$ has RREF

$$\begin{bmatrix} 1 & 0 & -23 & 16 & 0 \\ 0 & 1 & -10 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) find a basis for S

b) find a basis of S consisting of the original vectors

answer: $S = \text{row}(A)$

and a basis for $\text{Row}(A)$ is the set of non-zero rows in RREF of A

non-zero rows in RREF of A

a) basis = $\{ [1 \quad 0 \quad -23 \quad 16 \quad 0], [0 \quad 1 \quad -10 \quad 7 \quad 0], [0 \quad 0 \quad 0 \quad 0 \quad 1] \}$

b) what we want is to select 3 of the 4 original rows, but we don't know which ones to pick

- but the method to find Col(A) gives the original columns in the answer, so use transpose

$$A^T = \begin{array}{cccc} \vec{v}_1^{\,T} & \vec{v}_2^{\,T} & \vec{v}_3 & \vec{v}_4 \\ \end{array}$$
$$A^T = \begin{bmatrix} 1 & -3 & 2 & -3 \\ -2 & 7 & -5 & 6 \\ -3 & -1 & 4 & 9 \\ 2 & 1 & -3 & -6 \\ -4 & -3 & 7 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$S = Col(A^T)$

the basis we want is $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$

section 3.5: cont'd 2022/10/28

theorem: let S be a subspace of $\mathbb{R}^n$ and let $B = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ be a basis for S. then any vector $\vec{u}$ in S can be uniquely expressed as a linear combination

$$\vec{u} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots c_k \vec{v}_k$$

notation:

notation: $[\vec{u}]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_k \end{bmatrix}$

these are the coordinates of $\vec{u}$ with to B, but also the coefficients of the linear combo

if there isn't any subscript, then the default is $\{\hat{i}, \hat{j}, \hat{k}\}$ for $\mathbb{R}^3$

example: let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

a) show that $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of $\mathbb{R}^3$

b) find the coordinates of $\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ with respect to B.

answer: a) a basis of $\mathbb{R}^3$ requires 3 LI vectors

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

no free variables

$\therefore$ LI

and B is a basis for $\mathbb{R}^3$

b) want coords of u with respect to B

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 0 & 1 & 3 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 0 & 1 & 3 \\ 0 & 3 & 1 & 4 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 13 \end{array}\right]$$

so $\vec{u} = -5\vec{v}_1 - 3\vec{v}_2 + 13\vec{v}_3$

$$\vec{u} = \begin{bmatrix} -5 \\ -3 \\ 13 \end{bmatrix}_B$$