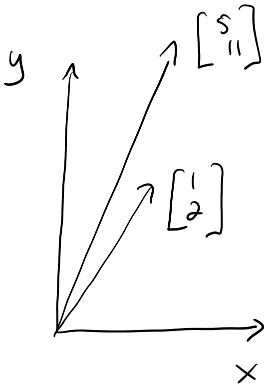
Section 3.6: Intro to Linear Transformations

Friday, October 28, 2022 12:57 PM

example: (1) let $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

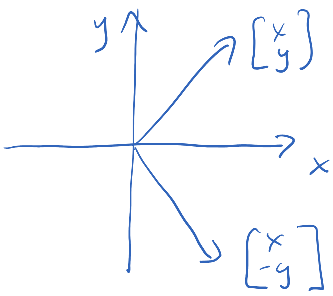
then $A\vec{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$



matrix A transformed $\vec{x}$ by rotating it and scaling it

(2) Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

then $A\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$



this A reflects the vector $\vec{x}$ through the x-axis

notation: if $f(x) = x^2 + 1$, then we say that

$$f : \mathbb{R} \to \mathbb{R}$$

set of inputs in $\mathbb{R}$

the set of outputs is also $\mathbb{R}$

domain : $\mathbb{R}$ inputs of f

codomain : $\mathbb{R}$ general set of outputs range : $[1, \infty)$ set of all possible outputs of f

In general, if A is an $M \times N$ matrix

$$\underbrace{T}_{\substack{\text{transformation} \\ T}}(\vec{x}) = \underbrace{A}_{M\times N} \underbrace{\vec{x}}_{N\times 1}$$

then $T: \mathbb{R}^N \to \mathbb{R}^M$

$\vec{x}$ is $N \times 1$ $A\vec{x}$ is $M \times 1$

definition : a transformation $T: \mathbb{R}^N \to \mathbb{R}^M$ is linear if

① $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for $\vec{u}, \vec{v}$ in $\mathbb{R}^N$

② $T(c\vec{v}) = c\,T(\vec{v})$ for all $\vec{v}$ in $\mathbb{R}^N$ and all scalars c

if T is linear transformation, then there is a matrix A such that

$$T(\vec{x}) = A\vec{x}$$

same transform

to find A, we find the columns of A by looking at what T does to each of the basis vectors in $\mathbb{R}^N$

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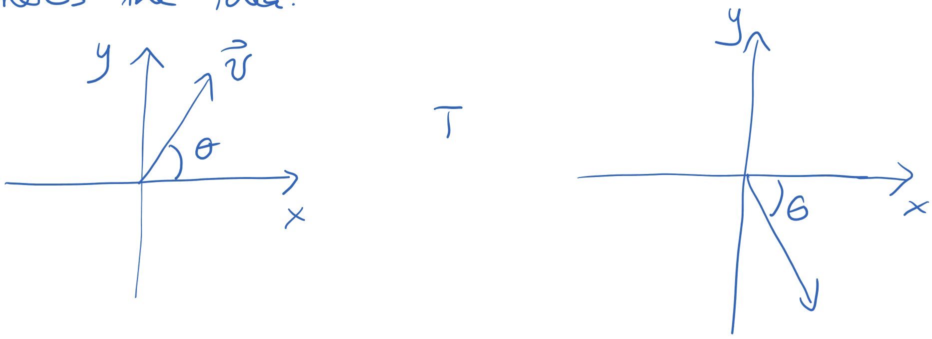
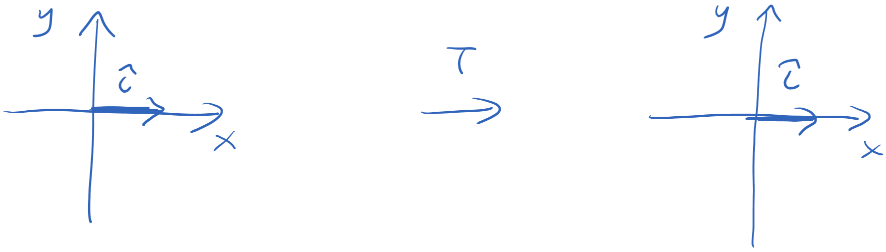
section 3.6 : cont'd 2022/10/31

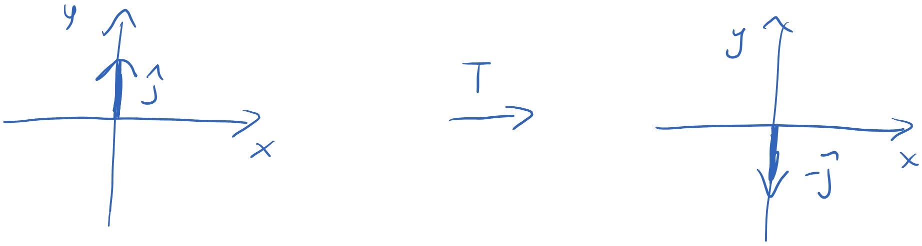
example : let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that reflects a vector $\vec{v}$ through the x-axis. Find the matrix of this transformation.

answer: here's the idea:



but how to find the matrix? consider the following: (transform basis vectors $\hat{i}$ and $\hat{j}$)





so $T(\hat{i}) = \hat{i}$ $\qquad$ $T(\hat{j}) = -\hat{j}$

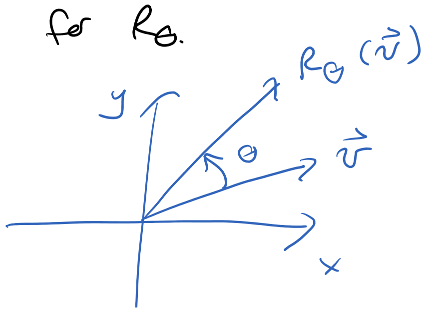
first column of A $\qquad$ second column of A

then $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

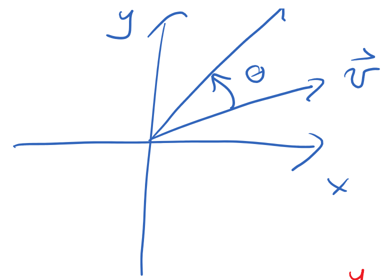
example: consider the transformation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates a vector $\vec{v}$ in $\mathbb{R}^2$ counterclockwise by an angle θ.

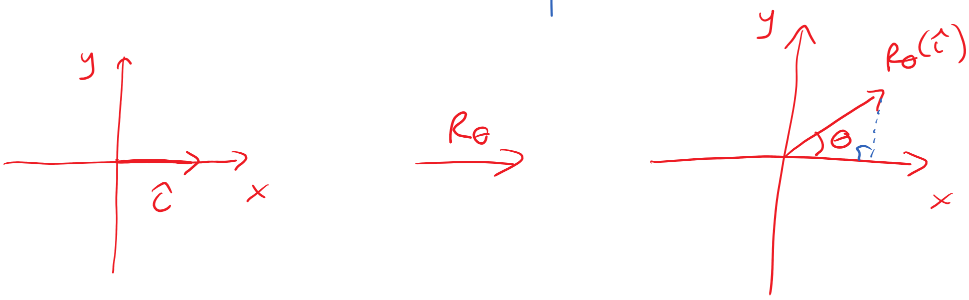
Find the matrix for R_θ.

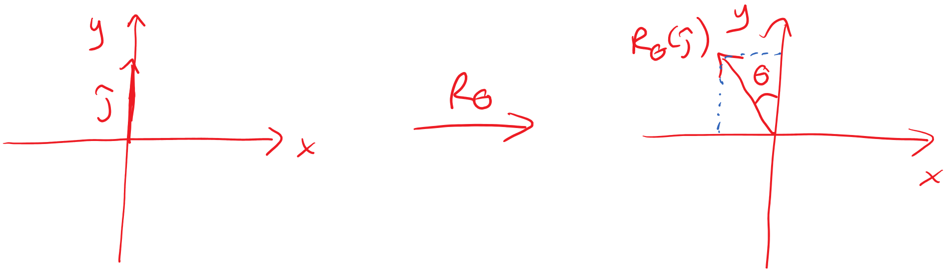
answer: the idea :



answer: the idea:







$$R_\theta \hat{i} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad R_\theta \hat{j} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

so the full rotation matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

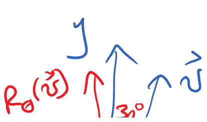
note: if you need this on a test I will provide it <u>unless</u> I ask you to derive it as we did here

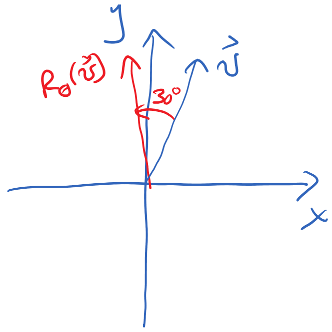
example: Rotate $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by $30°$ counterclockwise

answer: $$R_{30°}(\vec{v}) = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 - 1 \\ 1/2 + \sqrt{3} \end{bmatrix} \approx \begin{bmatrix} -0.13 \\ 2.23 \end{bmatrix}$$





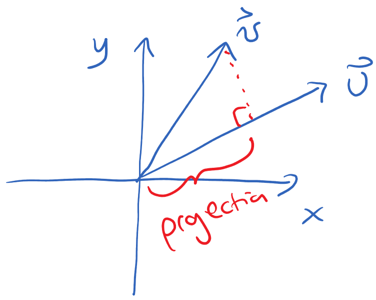
example: Find the matrix corresponding to projecting vectors onto $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$

note: in a test situation, I would not leave $\vec{u}$ as a generic vector – I'd give you values for a and b to reduce the difficulty

answer: $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T(\vec{v}) = \text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

the main idea:



let's find the components:

$\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\quad \hat{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$T(\hat{\imath}) = \text{proj}_{\vec{u}}(\hat{\imath})$$

$$= \frac{\vec{u}\cdot\hat{\imath}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

$$= \frac{a}{a^2+b^2}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{a^2+b^2}\begin{bmatrix} a^2 \\ ab \end{bmatrix}$$

$\hat{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T(\hat{\jmath}) = \text{proj}_{\vec{u}}(\hat{\jmath})$$

$$= \frac{\vec{u}\cdot\hat{\jmath}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

$$= \frac{\vec{u} \cdot \vec{y}}{\vec{u} \cdot \vec{u}} \vec{u}$$

$$= \frac{b}{a^2+b^2} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{a^2+b^2} \begin{bmatrix} ab \\ b^2 \end{bmatrix}$$

then $A = \frac{1}{a^2+b^2} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$

note: lets find the determinant of A:

$$\det(A) = \frac{1}{a^2+b^2} \left(a^2 b^2 - (ab)^2 \right) = 0$$

this matrix has no inverse - you cannot undo a projection

but: $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\begin{aligned} \det(R_\theta) &= \cos^2\theta - (-\sin^2\theta) \\ &= \cos^2\theta + \sin^2\theta \\ &= 1 \end{aligned}$$

so R_θ has an inverse and can be undone (which is to rotate clockwise by same angle)

A transformation is reversible $\iff$ Its matrix is invertible $\iff$ $\det(\text{matrix}) \neq 0$

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section 3.6 cont'd 2022/11/01

composition of transformations:

- what happens when we do multiple transformations one after another
  (rotate then reflect in y-axis)

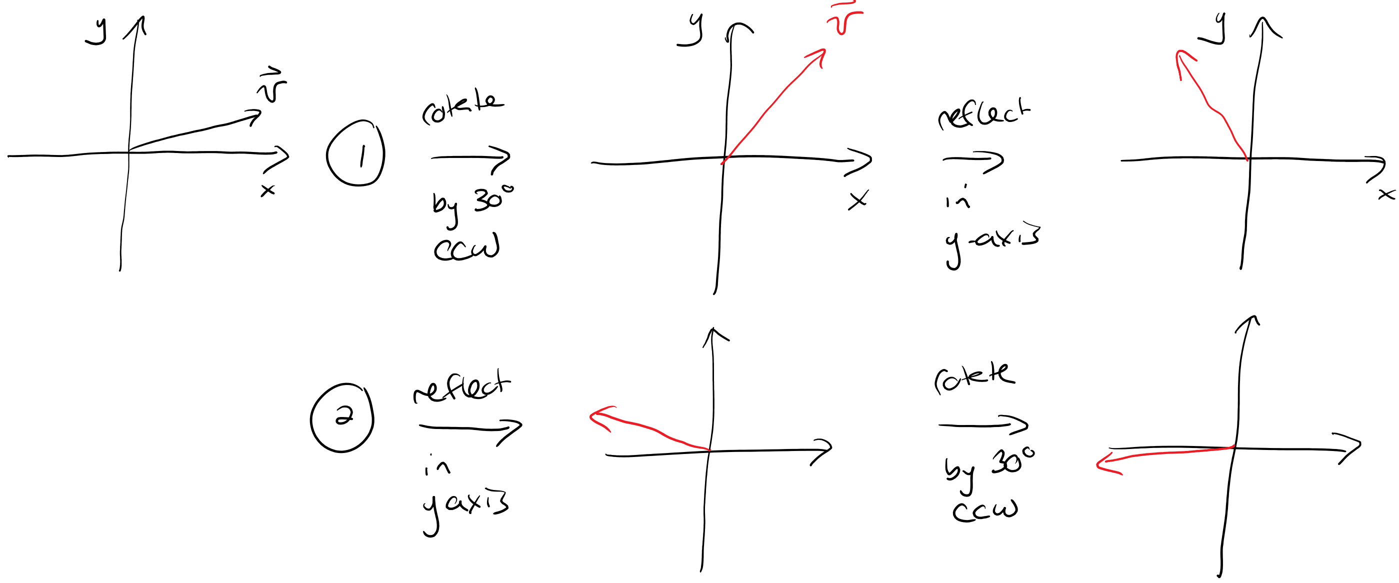
if we do transformation T first (associated matrix is A) then do transformation S next (associated matrix is B)

then we can write:

$$(S \circ T)(\vec{v}) = B A \vec{v}$$

this transformation is done first

note: why does order matter?



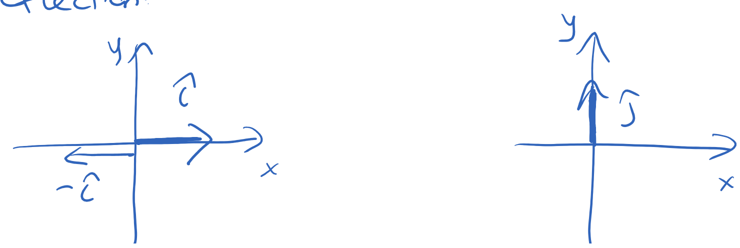
example: Find the matrix corresponding to rotating counterclockwise by 30° and then reflecting through y-axis.

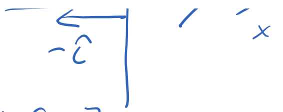
answer:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{30^\circ} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

reflection:





get $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

get $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

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order matters:

$$\underbrace{A R_{30^\circ}}_{\text{build in reverse order, starting from right}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$= \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

one last note: transformations are <u>linear</u>

so if $\vec{v} = \vec{v}_1 + \vec{v}_2$

then $T(\vec{v}) = T(\vec{v}_1 + \vec{v}_2)$

$$= T(\vec{v}_1) + T(\vec{v}_2)$$

example: Find $T(\vec{v})$ if $\vec{v} = 3\vec{v}_1 - 4\vec{v}_2$

and $T(\vec{v}_1) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $T(\vec{v}_2) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

answer: $T(\vec{v}) = T(3\vec{v}_1 - 4\vec{v}_2)$

$$= 3T(\vec{v}_1) - 4T(\vec{v}_2)$$

$$= 3\begin{bmatrix} 3 \\ 2 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -14 \end{bmatrix}$$