Section 4.1: Intro to Eigenvectors and Eigenvalues

Monday, November 07, 2022 10:13 AM

let A be a 2×2 matrix

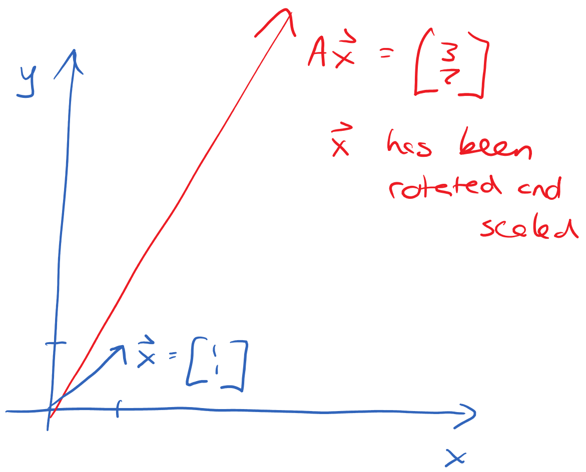
typically $A\vec{x}$ scales and rotates $\vec{x}$

example:

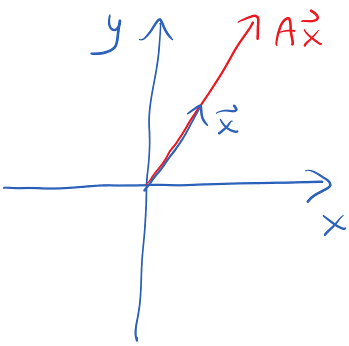
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

then $A\vec{x} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

Section 4.1 cont'd 2022/11/08



we are interested in the special vectors that are simply scaled by A and <u>not</u> rotated:



so $A\vec{x}$ is a multiple of $\vec{x}$

definition: let A be an $N \times N$ matrix

a scalar λ is called an eigenvalue
"lambda"

if there is a non-zero vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}$$

A is a matrix

λ is a scalar (constant)

and we also say that $\vec{x}$ is the eigenvector corresponding to λ.

example: $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. Show that $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors of A and find their associated eigenvalues.

answer: $A\vec{x}_1 = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\vec{x}_1$

$\lambda_1 = 2$

$A\vec{x}_2 = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\vec{x}_2$

$\lambda_2 = 3$

but how can you find λ if you don't know $\vec{x}$?

$$A\vec{x} = \lambda\vec{x}$$

$$A\vec{x} = \lambda I_2 \vec{x}$$

$$A\vec{x} - \lambda I_2 \vec{x} = 0$$

$$\underbrace{(A - \lambda I_2)}\vec{x} = 0$$

we don't want to be able to find a unique solution and reduce this $(A-\lambda I_2)$ to the identity matrix because the only solution will be $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

and we are interested in non-zero vectors

so we want $A-\lambda I_2$ to <u>not</u> have an inverse - we want the determinant to equal zero

so $\det(A-\lambda I_2) = 0$

this gives us a polynomial in λ whose roots are the eigenvalues of A

example: Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

answer:

$$A - \lambda I_2 = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix}$$

$$\det(A-\lambda I_2) = (1-\lambda)(4-\lambda) - 2(-1) = 0$$

$$4 - 5\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 3)(\lambda - 2) = 0$$

$$\lambda = 3, 2$$

note: a 2×2 matrix can have

1) 2 distinct real eigenvalues,
2) 1 real repeated eigenvalue
   (polynomial was a perfect square)
3) 2 complex eigenvalues

In general, for an $N\times N$ matrix A, we can find its eigenvalues by solving

$$\det(A - \lambda I_N) = 0$$

and the associated eigenvectors are found by solving

$$(A - \lambda I_N)\vec{x} = \vec{0}$$

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example: now find the eigenvectors for the eigenvalues in the previous example

note: we found that $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

has eigenvalues
$\lambda_1 = 2, \quad \lambda_2 = 3$

answer: now solve $(A - \lambda I)\vec{x} = 0$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix}$$

so use $\lambda_1 = 2$: $(A - \lambda I)\vec{x} = 0$

$$\begin{bmatrix} -1 & 2 \\ \end{bmatrix}\begin{bmatrix} x \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \rightsquigarrow \begin{array}{cc} x & y \\ \end{array}\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

y free variable

let $y = t$

$$\begin{cases} x = 2t \\ y = t \end{cases}$$

$$\vec{x}_1 = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

so $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector, $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$, all multiples are eigenvectors

optional check: $A\vec{x} = \lambda\vec{x}$

$$\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A\vec{x}_1 \qquad\qquad = \lambda_1 \vec{x}_1 \quad \checkmark$$

so now do $\lambda_2 = 3$:

solve $(A - \lambda I)\vec{x} = 0$

$$\begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{array}{cc} x & y \\ \end{array}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

let $y = t$ (free variable)

let $y = t$ (free variable)

$$\begin{cases} x = t \\ y = t \end{cases}$$

$$\vec{x}_2 = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

final answer: $\lambda_1 = 2$ has eigenvector $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$ has eigenvector $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

definition: eigenspace — the collection of all eigenvectors corresponding to a certain λ is called an eigenspace and is denoted by E_λ

so for previous example,

for $\lambda_1 = 2$, $E_{\lambda_1} = \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$

$\lambda_2 = 3$, $E_{\lambda_2} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

note: observe that if $\vec{x}$ is an eigenvector for a certain λ, then any multiple of $\vec{x}$ is also an eigenvector for that λ

also, if $\vec{x}_1$ and $\vec{x}_2$ are both eigenvectors corresponding to the same λ, then the vector $(\vec{x}_1 + \vec{x}_2)$ is also an eigenvector