Section 4.2: Determinants

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2×2 matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3×3 matrices:

we can calculate the determinant in the same way we did the cross product

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 5 \\ 3 & -1 & 2 \end{bmatrix} \begin{matrix} 2 & 1 \\ 4 & 0 \\ 3 & -1 \end{matrix}$$

$$\det(A) = 2(0)(2) + 1(5)(3) + 3(4)(-1)$$
$$- 3(0)(3) - 2(5)(-1) - 1(4)(2)$$
$$= 0 + 15 - 12 - 0 + 10 - 8$$
$$= 5$$

⇐ answer is a single number

the method of minors

- general case of how to calculate a determinant

start with a 3×3:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

first, assign the signs: $\begin{bmatrix} + & - & + \end{bmatrix}$

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then $\det(A) = +a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$+ a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

example: Find the determinant of

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 5 \\ 3 & -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

using the method of minors.

a) across Row 1 :

$$\det(A) = +2\begin{vmatrix} 0 & 5 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 0 \\ 3 & -1 \end{vmatrix}$$

$$= 2(0+5) - 1(8-15) + 3(-4-0)$$

$$= 10 + 7 - 12$$
$$= 5$$

b) across Row 2 :

$$\det(A) = -4\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} + 0\begin{vmatrix} \text{don't} \\ \text{care} \end{vmatrix} - 5\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= -4(2+3) + 0 - 5(-2-3)$$

$$= 5$$

note: could expand across Column 2 instead example: Find the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 6 & 0 \\ 2 & 1 & 5 & 6 \\ 3 & 4 & -2 & 7 \end{bmatrix} \qquad \begin{bmatrix} + & & & \\ - & + & - & + \\ & & & \\ & & & \end{bmatrix}$$

$$\det(A) = -0 + 0 - 6\begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 6 \\ 3 & 4 & 7 \end{vmatrix} + 0$$

$$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 6 \\ 3 & 4 & 7 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{matrix} \quad = 7 + 36 + 32 - 12 - 24 - 28 = 11$$

$$\det(A) = -6(11) = -66$$

Section 4.2: Cont'd

special case: triangular matrices

example: find determinant of $A = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 0 & 5 & 1 & 6 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

answer: expand about column 1

$$\det(A) = +2\begin{vmatrix} 5 & 1 & 6 \\ 0 & 4 & 5 \\ 0 & 0 & 2 \end{vmatrix} \begin{matrix} 5 & 1 \\ 0 & 4 \\ 0 & 0 \end{matrix}$$

$= 2(5)(4)(2)$ because of all the zeros

In general, the determinant of an upper or lower triangular matrix is the product of the main diagonal

Properties of Determinants:

1. $\det(A^T) = \det(A)$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2. If A has a row or column with all zero entries, then $\det(A) = 0$.

3. If $A \xrightarrow[R_i \Leftrightarrow R_j]{} B$ (row swap operation)

   then

   $\det(A) = -\det(B)$

   $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det(A) = -2$

   $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ $\det(B) = 2$

4. If $A \xrightarrow{kR_i} B$, (multiply row by constant)

   then $k\det(A) = \det(B)$ *

   $C = \begin{bmatrix} 5 & 10 \\ 3 & 4 \end{bmatrix}$ ← mult by 5

   * also true if you multiply a column by k

5. If $A \xrightarrow{R_i + kR_j} B$ (add multiple of another row)

   then $\det(A) = \det(B)$

6. If A has 2 identical rows or columns,

then

$$\det(A) = 0$$

example: Find determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 5 & 1 \end{bmatrix}$

$\downarrow R_2 - 2R_1$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix}$

$\det(B) = 0$

but $R_2 - 2R_1$ doesn't change the determinant, so $\det(A) = 0$

further properties of determinants:

if A and B are $N \times N$ matrices and k is a scalar, then

1. $\det(AB) = \det(A)\det(B)$
2. $\det(kA) = k^N \det(A)$ $\leftarrow$ know this one
3. $\det(A^{-1}) = \dfrac{1}{\det(A)}$

inverse of a matrix: cofactor method

example: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$ using the cofactor method.

answer: $C = \Big[+\begin{vmatrix} 3 & 4 \\ & \end{vmatrix} \quad \begin{vmatrix} 2 & \\ & \end{vmatrix} \quad +\begin{vmatrix} & \\ & \end{vmatrix} \Big]$

answer: $C = \begin{bmatrix} +\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{bmatrix}$

$$= \begin{bmatrix} 1 & -10 & 7 \\ 1 & 4 & -3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} C^T$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{bmatrix}$$

application of determinants: <u>Cramer's Rule</u>

notation: if $A\vec{x} = \vec{b}$, then $A_i(\vec{b})$ is the matrix in which column i is replaced by $\vec{b}$

Cramer's rule: If A is $N \times N$ and $A\vec{x} = \vec{b}$,

then $x_i = \dfrac{\det(A_i(\vec{b}))}{\det(A)}$

example: For the following system, solve for y

Using Cramer's Rule.

$$\begin{cases} x + 2y - z = 2 \\ 3x + 7y - 5z = 5 \\ -x - 2y = 1 \end{cases}$$

answer:

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -5 \\ -1 & -2 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}}_{\vec{b}}$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -5 \\ -1 & -2 & 0 \end{bmatrix} \qquad \det(A) = -1$$

$$A_2(\vec{b}) = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & -5 \\ -1 & 1 & 0 \end{bmatrix} \qquad \det(A_2(\vec{b})) = 7$$

$$y = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{7}{-1} = -7$$