Section 4.3: Eigenvalues and Eigenvectors

Monday, November 14, 2022 9:13 AM

3×3 case:

example: let $A = \begin{bmatrix} -4 & 0 & 3 \\ 0 & 2 & 0 \\ -6 & 0 & 5 \end{bmatrix}$

Find all eigenspaces of A.

answer: step 1: find eigenvalues λ using $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ -6 & 0 & 5-\lambda \end{vmatrix} \begin{matrix} -4-\lambda & 0 \\ 0 & 2-\lambda \\ -6 & 0 \end{matrix}$$

$$0 = (-4-\lambda)(2-\lambda)(5-\lambda) + 18(2-\lambda)$$

$$0 = (2-\lambda)\left[(-4-\lambda)(5-\lambda) + 18\right]$$

$$0 = (2-\lambda)\left[-20 - \lambda + \lambda^2 + 18\right]$$

$$0 = (2-\lambda)(\lambda^2 - \lambda - 2)$$

$$0 = (2-\lambda)(\lambda - 2)(\lambda + 1)$$

$$\lambda = \underbrace{2, 2}_{\text{multiplicity } 2}, \underbrace{-1}_{\text{multiplicity } 1}$$

(number of times the value occurs in the solution)

step 2: find eigenvectors using $(A - \lambda I)\vec{x} = 0$

for $\lambda_1 = 2$

$$(A - \lambda I)\vec{x} = 0$$

$$\begin{bmatrix} -4-\lambda_1 & 0 & 3 \\ 0 & 2-\lambda_1 & 0 \\ -6 & 0 & 5-\lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -6 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & 0 & 3 & 0 \end{array}\right]$$

$\Big\downarrow$ RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$y = s \qquad z = t \qquad$ free variables

$$x - \tfrac{1}{2} z = 0$$

$$\begin{cases} x = \frac{1}{2} t \\ y = s \\ z = t \end{cases} \qquad \text{and} \qquad \vec{x} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

now do $\lambda_2 = -1$:

solve $(A - \lambda_2 I)\vec{x} = 0$

$$\left[\begin{array}{ccc|c} -4-\lambda_2 & 0 & 3 & 0 \\ 0 & 2-\lambda_2 & 0 & 0 \\ -6 & 0 & 5-\lambda_2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ -6 & 0 & 6 & 0 \end{array}\right]$$

$\Big\downarrow$ RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable $z = t$

$$x - z = 0$$
$$y = 0$$

$$\begin{cases} x = t \\ y = 0 \\ z = t \end{cases} \quad \text{so} \quad \vec{x}_2 = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

step 3: write out eigenspaces

for $\lambda_1 = 2$, $E_{\lambda_1} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \right)$

$\lambda_2 = -1$, $E_{\lambda_2} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$

Algebraic multiplicity of an eigenvalue is the number of times the eigenvalue is repeated as a root of

$$\det(A - \lambda I) = 0$$

example: if $\det(A - \lambda I) = (\lambda + 1)^3 (\lambda - 2)(\lambda - 4)^2$

eigenvalue	algebraic multiplicity
-1	3
2	1
4	2

Geometric multiplicity of an eigenvalue is the dimension of E_λ (number of vectors in the basis)

example: in the previous 3x3 example, we had example: in the previous 3x3 example, we had

for $\lambda_1 = 2$, $E_{\lambda_1} = \text{span}\left(\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

so $\dim(E_\lambda) = 2$ (2D space)

geometric multiplicity of $\lambda_1 = 2$

- the algebraic and geometric multiplicity of an eigenvalue can be different, but

$$\text{geo mult} \leq \text{algebraic mult}$$

(if geo < alg, we say matrix is defective)

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example: Find all eigenvalues and eigenvectors for the following matrix A. Then give the algebraic and geometric multiplicities for each eigenvalue.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

answer: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(1-\lambda)(2-\lambda) = 0$$

$$\lambda = 1, \ 2$$

alg mult of 2

alg mult of 1

so $\lambda_1 = 1$: solve $(A - \lambda_1 I)\vec{x} = 0$

$$\left[\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable so $x = t$
$y = 0$
$z = 0$

$$\begin{cases} x = t \\ y = 0 \\ z = 0 \end{cases} \quad \text{and} \quad \vec{x}_1 = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

geo mult of λ_1 is 1

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and $\lambda_2 = 2$

$$(A - \lambda I)\vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free var

$x - 5z = 0$

$y - 3z = 0$

let $z = t$

$$\begin{cases} x = 5t \\ y = 3t \\ z = t \end{cases} \quad \text{and} \quad \vec{x}_2 = t\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

final answer:

recall
geo $\leq$ alg

eigenvalue	eigenvector(s)	alg mult	geo mult
1	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	2	1
2	$\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$	1	1

for $A\vec{x} = \lambda \vec{x}$:

$$A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda^2\vec{x}$$
$$A^3\vec{x} = A(A^2\vec{x}) = A(\lambda^2\vec{x}) = \lambda^2(A\vec{x}) = \lambda^3\vec{x}$$

and $A^N\vec{x} = \lambda^N\vec{x}$ for any $N \geq 1$

example: given $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$,

find $A^{100}\vec{v}$.

answer: find eigenvalues and eigenvectors of A

note: we previously found that for this A,

eigenvalue	eigenvector
$\lambda_1 = 2$	$\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
$\lambda_2 = 3$	$\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and we can see that $\vec{x}_1$ and $\vec{x}_2$ are LI (linearly independent)
so the set $\{\vec{x}_1, \vec{x}_2\}$ forms a basis for $\mathbb{R}^2$

useful fact: eigenvectors obtained from distinct eigenvalues are <u>always</u> LI

$$\begin{bmatrix} 7 \\ 3 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 7 \\ 1 & 1 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \end{array}\right]$$

so $c_1 = 4$
$c_2 = -1$

$$\vec{v} = 4\vec{x}_1 - \vec{x}_2$$

now sub into $A^{100}\vec{v}$:

$$A^{100}\vec{v} = A^{100}(4\vec{x}_1 - \vec{x}_2)$$
$$= 4A^{100}\vec{x}_1 - A^{100}\vec{x}_2$$
$$= 4\lambda_1^{100}\vec{x}_1 - \lambda_2^{100}\vec{x}_2$$
$$= 4\cdot 2^{100}\begin{bmatrix}2\\1\end{bmatrix} - 3^{100}\begin{bmatrix}1\\1\end{bmatrix}$$
$$= \begin{bmatrix}(2^{103} - 3^{100})\\(2^{102} - 3^{100})\end{bmatrix}$$

single number

this is a 2×1

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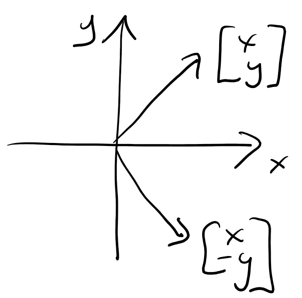
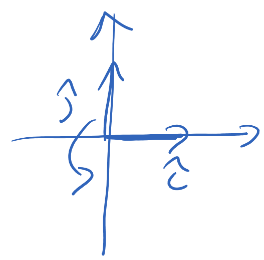
note on geometry of eigenvectors:

recall the reflection matrix $R = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$



this matrix represents a reflection through the x-axis



$R\hat{i} = 1\cdot\hat{i}$ — $\hat{i}$ is an eigenvector for $\lambda_1 = 1$

$R\hat{j} = -1\,\hat{j}$ — $\hat{j}$ is an eigenvector for $\lambda_2 = -1$

what about the rotation matrix R_θ which rotates a vector counterclockwise through angle θ?

- will not have <u>real</u> eigenvalues because $\vec{v}$ and $R_\theta(\vec{v})$ are not in the same direction

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<u>complex eigenvalues:</u>

example: find the eigenvalues for

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

answer:

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-1-\lambda) - (-2)(4) = 0$$

$$-3 + \lambda - 3\lambda + \lambda^2 + 8 = 0$$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{2 \pm 4i}{2} = 1 \pm 2i$$

note: if matrix A has real entries, complex eigenvalues always occur in conjugate pairs

now let's find eigenvectors:

$\lambda_1 = 1 + 2i$

$A = \begin{bmatrix} 3 & -2 \end{bmatrix}$

$\lambda_1 = 1 + 2i$ $\quad$ $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

solve $(A - \lambda I)\vec{x} = 0$

$$\left[\begin{array}{cc|c} 2-2i & -2 & 0 \\ 4 & -2-2i & 0 \end{array}\right]$$

method #1: $\downarrow \frac{1}{2-2i} R_1$ $\quad$ * could instead multiply by $(2+2i)R_1$ instead

$$\left[\begin{array}{cc|c} 1 & \frac{-2}{2-2i} & 0 \\ 4 & -2-2i & 0 \end{array}\right]$$

but what is $\frac{-2}{2-2i}$?

$$\frac{-2}{2-2i} = \frac{-2}{2-2i}\left(\frac{2+2i}{2+2i}\right) = \frac{-4-4i}{4-4i^2}$$

$$= \frac{-4-4i}{8}$$

$$= -\tfrac{1}{2} - \tfrac{1}{2}i$$

so augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2}-\frac{1}{2}i & 0 \\ 4 & -2-2i & 0 \end{array}\right]$$

$$\downarrow R_2 - 4R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2}-\frac{1}{2}i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

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method #2 for finding the RREF:

$$\left[\begin{array}{cc|c} 2-2i & -2 & 0 \\ 4 & -2-2i & 0 \end{array}\right]$$

$\downarrow$ $R_1 \Leftrightarrow R_2$

$$\left[\begin{array}{cc|c} 4 & -2-2i & 0 \\ 2-2i & -2 & 0 \end{array}\right]$$

if we have done this correctly, the second row will <u>always</u> be a multiple of the first

$\downarrow$ $\frac{1}{4}R_1$, plus some vigorous handwaving

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2}-\frac{1}{2}i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

free variable so let $y=t$

$x + (-\frac{1}{2} - \frac{1}{2}i)y = 0$

now find the eigen vector:

$$\begin{cases} x = (\frac{1}{2} + \frac{1}{2}i)t \\ y = t \end{cases}$$

$$\vec{x}_1 = t\begin{bmatrix} \frac{1}{2} + \frac{1}{2}i \\ 1 \end{bmatrix}$$

$$= t\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$ if you prefer

notice that we have only found

$\lambda_1 = 1+2i$ has $\vec{x}_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

but we will find that the complex conjugate of λ_1 will have the complex conjugate of $\vec{x}_1$ as the eigenvector (you can just state this on tests, though I will demonstrate it below)

$\lambda_2 = 1-2i$ has $\vec{x}_2 = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$

why? $\lambda_2 = 1 - 2i$

augmented matrix is $\left[\begin{array}{cc|c} 2+2i & -2 & 0 \\ 4 & -2+2i & 0 \end{array}\right]$

$\rightsquigarrow R_1 \Leftrightarrow R_2$

$\left[\begin{array}{cc|c} 4 & -2+2i & 0 \\ 2+2i & -2 & 0 \end{array}\right]$

$\rightsquigarrow \frac{1}{4} R_1$

$\left[\begin{array}{cc|c} 1 & -\frac{1}{2}+\frac{1}{2}i & 0 \\ 0 & 0 & 0 \end{array}\right]$

complex conjugate of previous RREF

so will get complex conjugate of the previous eigenvector

so $\lambda_1 = 1+2i$ has $\vec{x}_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

$\lambda_2 = 1-2i$ has $\vec{x}_2 = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$

or $\lambda_{1,2} = 1 \pm 2i$ has $\vec{x}_{1,2} = \begin{bmatrix} 1 \pm i \\ 2 \end{bmatrix}$

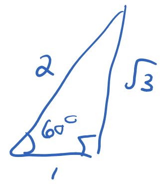
and if you want to be really terse, notice that you can write:

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}_{1,2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \pm i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

example: find the eigenvalues of $R(60°)$.

answer: $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



$$R(60°) = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$\det(A - \lambda I) = 0 = \begin{vmatrix} (1/2 - \lambda) & -\sqrt{3}/2 \\ \sqrt{3}/2 & (1/2 - \lambda) \end{vmatrix}$$

$$0 = (1/2 - \lambda)(1/2 - \lambda) + 3/4$$

$$(1/2 - \lambda)^2 = -3/4$$

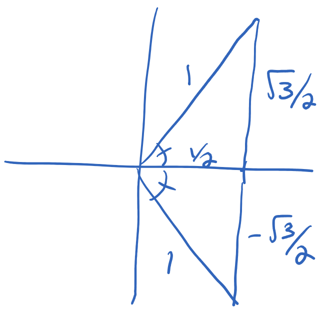
$$(1/2 - \lambda) = \pm\sqrt{-\tfrac{3}{4}}$$

$$1/2 - \lambda = \pm \frac{i\sqrt{3}}{2}$$

$$-\lambda = -1/2 \pm \frac{i\sqrt{3}}{2}$$

$$\lambda = 1/2 \pm \frac{i\sqrt{3}}{2} \quad \text{or} \quad \frac{1 \pm i\sqrt{3}}{2}$$

but in the complex plane, what do these eigenvalues look like?



the angles are 60°

and the eigenvalues are $1e^{\pm i\pi/3}$

in general, the eigenvalues of R_θ, the rotation matrix, are $e^{i\theta}$ and $e^{-i\theta}$